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## THEORY OF GROUND VIBRATIONS OF A TWO-BLADE HELICOPTER ROTOR ON ANISOTROPIC FLEXIBLE SUPPORTS

By Robert P. Coleman and Arnold M. Feingold

Langley Memorial Aeronautical Laboratory
Langley Field, Va.



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# THEORY OF GROUND VIBRATIONS OF A TWO-BLADE HELICOPTER ROTOR ON ANISOTROPIC FLEXIBLE SUPPORTS

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## SUMMARY

An extension of previous work on the theory of self-excited mechanical oscillations of hinged rotor blades has been made. Previously published papers cover the cases of three or more rotor blades on elastic supports (such as landing gear) having either equal or unequal support stiffness in different directions and the case of one or two blade rotors on supports having equal stiffness in all horizontal directions. The missing case of one or two blades on unequal supports has been treated.

The mathematical treatment of this case is considerably more complicated than the other cases because of the occurrence of differential equations with periodic coefficients. The characteristic frequencies are obtained from an infinite-order determinant. Recurrence relations and convergence factors are used in finding the roots of the infinite determinant.

The results show the existence of ranges of rotational speed at which instability occurs (changed somewhat in position and extent) similar to those possessed by the two-blade rotor on equal supports. In addition, the existence of an infinite number of instability ranges which occurred at low rotor speeds and which did not occur in the cases previously treated is shown.

Simplifications occur in the analysis for the special cases of infinite and zero stiffness in one of the axes. The case of infinite stiffness in one axis is also of special interest because it is mathematically equivalent to a counterrotating rotor system. A design chart for finding the position of the principal self-excited instability range for the case of infinite support stiffness in one direction is included for the convenience of designers. It is expected that designers will be able to obtain sufficiently accurate information by considering only the cases of infinite and zero support stiffness along one direction together with the cases treated previously.

## INTRODUCTION

It is known that rotating-wing aircraft may experience violent vibrations while the rotor is turning and the aircraft is on the ground. It has been found that these vibrations can be explained without considering aerodynamic effects and that they are due to mechanical coupling between horizontal hub displacements and blade oscillations in the plane of rotation. A theoretical analysis of this vibration problem is given in references 1 and 2. Reference 1 deals with rotors having three or more equal blades on general supports and reference 2 deals with two-blade rotors on supports having the same stiffness in all directions.

Although in actual two-blade rotary-wing aircraft, the stiffness of the supports along the longitudinal direction is certainly different from the lateral stiffness, the equality of the stiffnesses was assumed in reference 2 because it permitted the mathematical simplification of dealing with differential equations having constant coefficients and it was believed that a theory employing such an assumption would be sufficient to indicate the nature of the most violent types of ground instability.

The present paper gives a theoretical investigation of the general case of a two-blade rotor mounted upon supports of unequal stiffness along the two stationary principal axes. It thus generalizes the problem of reference 2, and rounds out the studies of ground resonance begun in reference 1. As was shown in reference 2, a two-blade rotor possesses different dynamic properties along and normal to the line of the blades. Equations of motion with constant coefficients for the problem treated in reference 2 could be obtained by using a coordinate system rotating with the rotor. This procedure succeeded because the supports were assumed isotropic (equal stiffness in all directions). When the supports are anisotropic, however, it is impossible to avoid the appearance of periodic coefficients in the equations of motion.

The present method of solving the differential equations of motion follows closely the process employed in reference 3 for a vibration problem in two degrees of freedom. The form of solution is expressed by an exponential factor times a complex Fourier series. Substitution of the formal solution into the equations of motion yields an infinite set of algebraic equations and an infinite-order determinant for the determination of the Fourier coefficients and the characteristic frequencies. The subsequent analysis is concerned with methods of finding the roots of the infinite determinant.

Because the methods herein employed may also be useful in other rotor problems, particularly in those involving forward-flight effects, the mathematical analysis is presented in some detail.

It is expected that designers will be able to obtain sufficiently accurate information by considering only the cases of infinite or zero support stiffness along one direction together with the cases of references 1 and 2. In order to avoid the necessity for extensive calculations, a design chart is included giving the location of the principal self-excited-instability range for the case of infinite support stiffness in one direction.

## DERIVATION OF THE EQUATIONS OF MOTION

The symbols used herein are defined in appendix A.

The equations of motion are obtained from Lagrange's equations and from the expressions for kinetic and potential energy. Four degrees of freedom of the rotor system are considered: components of deflection of the rotor hub in the plane of rotation, and hinge deflections of the two rotor blades about their vertical hinges. All motions are thus assumed to occur in the plane of the rotor. The rotor is assumed to rotate at a constant angular velocity $\omega$. The analysis can be applied to rotors without hinges by assuming an effective spring stiffness and hinge position to represent the elastic deflection of the blade.

The pertinent physical parameters are:

$a$ radial position of vertical hinge

$b$ distance from vertical hinge to center of mass of blade

$m_b$ mass of rotor blade

$m$ effective mass of rotor supports

$r$ radius of gyration of blade about center of mass

$K_x$, $K_y$ spring constants of the rotor supports along the X- and Y-directions, respectively

$K_\beta$ spring constant of hinge self-centering spring

Let the origin of the X,Y-coordinate system be placed at the undisturbed position of the rotor hub. At time $t$ equal to 0, the line through the blade hinges and rotor hub is assumed parallel to the X-axis. After a time interval $t$, let the rotor hub deflection be $z$ and hinge deflections be $\beta_1$ and $\beta_2$, respectively, where $z$ is the complex position coordinate measured in a coordinate system rotating with the rotor. (See fig. 1.) Then the positions of the centers of mass of the two blades, as measured in fixed coordinates, will be, respectively,

$$\begin{aligned} z_1 &= \left(z + a + be^{i\beta_1}\right) e^{i\omega t} \\ z_2 &= \left(z - a - be^{i\beta_2}\right) e^{i\omega t} \end{aligned} \tag{1}$$

The kinetic energy of the rotor system $T$ can be written as

$$\begin{aligned} T = \frac{1}{2}m_b &\left\{\dot{z}_1\dot{\bar{z}}_1 + \dot{z}_2\dot{\bar{z}}_2 + r^2\left[\left(\omega + \dot{\beta}_1\right)^2 + \left(\omega + \dot{\beta}_2\right)^2\right]\right\} \\ &+ \frac{1}{2}m(\dot{z} + i\omega z)(\dot{\bar{z}} - i\omega\bar{z}) \end{aligned} \tag{2}$$

The first term in equation (2) represents the kinetic energy of the rotor blades, including the energy due to rotation, and the second term is the contribution of the rotor hub.

Upon expanding equation (1) into power series in $\beta_1$ and $\beta_2$ (only small deflections from equilibrium being considered) and substituting into equation (2), there is obtained

$$\begin{aligned} T = &\frac{1}{2}M(\dot{z} + i\omega z)(\dot{\bar{z}} - i\omega\bar{z}) \\ &+ \frac{1}{2}m_b\left[-\left(\beta_1 - \beta_2\right)\left(i\omega^2 bz - i\omega^2 b\bar{z} + \omega b\dot{z} + \omega b\dot{\bar{z}}\right)\right. \\ &+ \left(\dot{\beta}_1 - \dot{\beta}_2\right)\left(bi\dot{\bar{z}} + \omega bz + \omega b\bar{z} - bi\dot{z}\right) \\ &\left.+ \left(b^2 + r^2\right)\left(\dot{\beta}_1{}^2 + \dot{\beta}_2{}^2\right) - ab\omega^2\left(\beta_1{}^2 + \beta_2{}^2\right)\right] \end{aligned} \tag{3}$$

where only the terms that are quadratic in the variables have been retained, and $M$ represents the total mass of the rotor system.

The potential energy of the system $V$ is given by

$$V = \frac{K_\beta}{2}\left(\beta_1{}^2 + \beta_2{}^2\right) + \frac{K}{2} z\bar{z}$$

$$- \frac{\Delta K}{4}\left(z^2 e^{2i\omega t} + \bar{z}^2 e^{-2i\omega t}\right) \tag{4}$$

where $K$ is the average stiffness and $\Delta K$ is a measure of the difference of the two principal stiffnesses; that is,

$$K = \frac{K_y + K_x}{2}$$

$$\Delta K = \frac{K_y - K_x}{2}$$

As in references 1 and 2, simplifications in the analysis are introduced by replacing the hinge variables $\beta_1$ and $\beta_2$ by the new variables $\theta_0$ and $\theta_1$

$$\theta_0 = \frac{b}{2}\left(\beta_1 + \beta_2\right)$$

$$\theta_1 = \frac{b}{2}\left(\beta_1 - \beta_2\right)$$

In terms of these new variables the expressions (3) and (4) become, respectively,

$$T = \frac{M}{2}(\dot{z} + i\omega z)(\dot{\bar{z}} - i\omega\bar{z}) + m_b\Bigg[(\dot{\bar{z}} - i\omega\bar{z})\left(i\dot{\theta}_1 - \omega\theta_1\right)$$

$$- (\dot{z} + i\omega z)\left(i\dot{\theta}_1 + \omega\theta_1\right) + \left(1 + \frac{r^2}{b^2}\right)\left(\dot{\theta}_0{}^2 + \dot{\theta}_1{}^2\right)$$

$$- \frac{a}{b}\omega^2\left(\theta_0{}^2 + \theta_1{}^2\right)\Bigg] \tag{5}$$

and

$$V = \frac{K}{2} z\bar{z} + \frac{K_\beta}{b^2}\left(\theta_0{}^2 + \theta_1{}^2\right) - \frac{\Delta K}{4}\left(z^2 e^{2i\omega t} + \bar{z}^2 e^{-2i\omega t}\right) \tag{6}$$

By use of the Lagrangian form of the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

the following equations of motion for the rotor system are finally obtained:

$$\ddot{\theta}_0 + \left(\omega^2 \Lambda_1 + \Lambda_2\right) \theta_0 = 0 \tag{7}$$

$$(D + i\omega)^2 z + i\mu(D + i\omega)^2 \theta_1 + \frac{K}{M} z - \frac{\Delta K}{M} \bar{z} e^{-2i\omega t} = 0 \tag{8}$$

$$(D - i\omega)^2 \bar{z} - i\mu(D - i\omega)^2 \theta_1 + \frac{K}{M} \bar{z} - \frac{\Delta K}{M} z e^{2i\omega t} = 0 \tag{9}$$

$$(D + i\omega)^2 z - (D - i\omega)^2 \bar{z} + 2i\left(1 + \frac{r^2}{b^2}\right)\left(D^2 + \Lambda_1 \omega^2 + \Lambda_2\right)\theta_1 = 0 \tag{10}$$

where the notation

$$D = \frac{d}{dt}$$

has been used, and the following combinations of the original parameters have been introduced:

$$\mu = \frac{2m_b}{M}$$

$$\Lambda_1 = \frac{a}{b\left(1 + \frac{r^2}{b^2}\right)}$$

$$\Lambda_2 = \frac{K_\beta}{m_b b^2 \left(1 + \frac{r^2}{b^2}\right)}$$

Equation (7) can be solved independently of the others since it is an equation in $\theta_0$ alone. The equivalent equation appeared also in references 1 and 2, and its solution represents in-phase motion of the blades with no resultant reaction (except torsion) at the rotor hub. This motion will not be further considered herein.

The problem is thus resolved into the solution of the three simultaneous equations (8) to (10). It will be noted that the terms with periodic coefficients in equations (8) and (9) disappear if $\frac{\Delta K}{M} = 0$, that is, if $K_y = K_x$. Equations (8) and (9) are thus reduced to the problem treated in reference (2).

## FORM OF SOLUTION OF EQUATIONS OF MOTION

The equations of motion (equations (8) to (10)) are similar in mathematical properties to Mathieu's equation, which occurs in the analysis of vibrating systems of one degree of freedom with variable elasticity. (See reference 4.) A generalized form of Mathieu's equation was solved analytically by Hill. (See reference 5, pp. 413-417.) An extension of Hill's method has been applied in reference 3 to a problem involving two degrees of freedom, and a further development of the method of reference 3 is followed in the present paper.

Equations (8) to (10) constitute a system of linear differential equations with periodic coefficients. Three second-order equations possess six linearly independent solutions that, according to the Floquet theory (reference 5, p. 412), are of the form of an exponential factor times a periodic function of time. Particular solutions are of the form

$$\left.\begin{aligned} z &= e^{i\omega_a t}P(t) + e^{-i\bar{\omega}_a t}\bar{Q}(t) \\ \bar{z} &= e^{i\omega_a t}Q(t) + e^{-i\bar{\omega}_a t}\bar{P}(t) \\ \theta_1 &= e^{i\omega_a t}R(t) + e^{-i\bar{\omega}_a t}\bar{R}(t) \end{aligned}\right\} \qquad (11)$$

where $\omega_a$ is known as the characteristic exponent, and $P(t)$, $Q(t)$, and $R(t)$ are periodic functions of period $\pi/\omega$.

Since $P(t)$, $Q(t)$, and $R(t)$ are periodic functions of $t$, they can be represented by complex Fourier series, and equations (11) become

$$\left.\begin{aligned} z &= \sum_{-\infty}^{\infty} A_l\, e^{(2l\omega+\omega_a)it} + \sum_{-\infty}^{\infty} \bar{B}_l\, e^{-(2l\omega+\bar{\omega}_a)it} \\ \bar{z} &= \sum_{-\infty}^{\infty} B_l\, e^{(2l\omega+\omega_a)it} + \sum_{-\infty}^{\infty} \bar{A}_l\, e^{-(2l\omega+\bar{\omega}_a)it} \\ \theta_1 &= \sum_{-\infty}^{\infty} C_l\, e^{(2l\omega+\omega_a)it} + \sum_{-\infty}^{\infty} \bar{C}_l\, e^{-(2l\omega+\bar{\omega}_a)it} \end{aligned}\right\} \qquad (12)$$

where $A_l$, $B_l$, and $C_l$ are complex constants.

Equations (11) and (12) show that, when the rotor system is stable and $\omega_a$ is real, the motion not only is not simple harmonic as was the case in references 1 and 2, but, in general, is not even periodic. The motion can be said to consist of a fundamental frequency $\omega_a$ plus "harmonics" of frequency $\omega_a + 2l\omega$ where $l$ is any integer. From equations (12) it is seen that the value of $\omega_a$ is not uniquely determinate, since $\omega_a + 2l\omega$ also satisfies equations (12). (The imaginary part of $\omega_a$ is definite, however.) It can be shown furthermore, that, corresponding to each value of $\omega_a$, $-\omega_a$ is also a solution. Only those three values of $\omega_a$ for which the real parts lie between 0 and $\omega$ need therefore be considered. These values will be referred to hereinafter as the three "principal" values of $\omega_a$. These values of $\omega_a$ differing from the principal value by $2l\omega$, or having opposite sign, will be referred to as "harmonics" of the corresponding principal value.

Since $z$ has been defined as a position coordinate in a rotating frame of reference, the values of $\omega_a$ can be interpreted as the natural frequencies of the rotor system in rotating coordinates.

## SOLUTION OF EQUATIONS OF MOTION

### Determinantal Equation

If the formal solution (equation (12)) is combined with the equations of motion (equations (8) to (10)), and the coefficients of each exponential time factor is separately equated to zero, an infinite set of homogeneous equations is obtained. These equations can be separated into two independent sets. Each equation of one set is the conjugate of an equation of the other set, and only one set need be considered. Thus

$$\left\{\frac{K}{M} - \left[\omega_a + (2l + 1)\omega\right]^2\right\} A_l - \frac{\Delta K}{M} B_{l+1}$$

$$- \mu i\left[\omega_a + (2l + 1)\omega\right]^2 C_l = 0 \tag{13}$$

$$\left\{\frac{K}{M} - \left[\omega_a + (2l - 1)\omega\right]^2\right\} B_l - \frac{\Delta K}{M} A_{l-1}$$

$$+ \mu i\left[\omega_a + (2l - 1)\omega\right]^2 C_l = 0 \tag{14}$$

$$-\left[\omega_a+(2l + 1)\omega\right]^2 A_l + \left[\omega_a + (2l - 1)\omega\right]^2 B_l$$

$$+ 2i\left(1 + \frac{r^2}{b^2}\right)\left[-(\omega_a + 2l\omega)^2 + \Lambda_1\omega^2 + \Lambda_2\right]C_l = 0 \tag{15}$$

where $l$ takes on all integral values from $-\infty$ to $\infty$.

In order that the values of $A_l$, $B_l$, and $C_l$ not equal zero, the determinant of the coefficients of $A_l$, $B_l$, and $C_l$ must be zero. This determinantal equation is

$$
\begin{vmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_{-4,-4} & a_{-4,-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & a_{-3,-4} & a_{-3,-3} & a_{-3,-2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & a_{-2,-3} & a_{-2,-2} & a_{-2,-1} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & a_{-1,-2} & a_{-1,-1} & a_{-1,0} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & a_{0,-1} & a_{0,0} & a_{0,1} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & a_{3,2} & a_{3,3} & a_{3,4} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{4,3} & a_{4,4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{vmatrix}
$$

$$= \Delta(\omega_a) = 0 \tag{16}$$

where

$$a_{-4,-4} = -1 + \frac{K/M}{(\omega_a - 3\omega)^2} \qquad a_{0,0} = -1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{{\omega_a}^2}$$

$$a_{-4,-3} = \Lambda_3 \qquad a_{0,1} = \left(\frac{\omega_a + \omega}{\omega_a}\right)^2$$

$$a_{-3,-4} = \left(\frac{\omega_a - 3\omega}{\omega_a - 2\omega}\right)^2 \qquad a_{1,0} = \Lambda_3$$

$$a_{-3,-3} = -1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{(\omega_a - 2\omega)^2} \qquad a_{1,1} = -1 + \frac{K/M}{(\omega_a + \omega)^2}$$

$$a_{-3,-2} = \left(\frac{\omega_a - \omega}{\omega_a - 2\omega}\right)^2 \qquad a_{1,2} = \frac{\Delta\ K/M}{(\omega_a + \omega)^2}$$

$$a_{-2,-3} = \Lambda_3 \qquad a_{2,1} = \frac{\Delta\ K/M}{(\omega_a + \omega)^2}$$

$$a_{-2,-2} = -1 + \frac{K/M}{(\omega_a - \omega)^2} \qquad a_{2,2} = -1 + \frac{K/M}{(\omega_a + \omega)^2}$$

$$a_{-2,-1} = \frac{\Delta\ K/M}{(\omega_a - \omega)^2} \qquad a_{2,3} = \Lambda_3$$

$$a_{-1,-2} = \frac{\Delta\ K/M}{(\omega_a - \omega)^2} \qquad a_{3,2} = \left(\frac{\omega_a + \omega}{\omega_a + 2\omega}\right)^2$$

$$a_{-1,-1} = -1 + \frac{K/M}{(\omega_a - \omega)^2} \qquad a_{3,3} = -1 + \frac{\Lambda_1\ \omega^2 + \Lambda_2}{(\omega_a + 2\omega)^2}$$

$$a_{-1,0} = \Lambda_3 \qquad a_{3,4} = \left(\frac{\omega_a + 3\omega}{\omega_a + 2\omega}\right)^2$$

$$a_{0,-1} = \left(\frac{\omega_a - \omega}{\omega_a}\right)^2 \qquad a_{4,3} = \Lambda_3$$

$$a_{4,4} = -1 + \frac{K/M}{(\omega_a + 3\omega)^2}$$

The determinant has been somewhat simplified by multiplying and dividing the rows and columns by various quantities, and the parameter $\Lambda_3$ has been substituted for its equivalent $\dfrac{\mu}{2\left(1 + r^2/b^2\right)}$.

Let this infinite determinant be $\Delta(\omega_a)$. The problem consists in solving the equation $\Delta(\omega_a) = 0$ for its roots $\omega_a$. These roots will be infinite in number, consisting of the three principal values of $\omega_a$, plus all their harmonics. The values of $\omega_a/\sqrt{K/M}$, as a function of $\omega/\sqrt{K/M}$, are seen to depend only on the values of the three nondimensional parameters $\Lambda_1, \frac{\Lambda_2}{K/M}$, and $\Lambda_3$, and the stiffness ratio parameter $\Delta K/K$.

A determinant of infinite order has meaning only insofar as it is defined as the limit of a determinant of finite order. Define $\Delta_n(\omega_a)$ as the determinant of order $6n - 3$ formed from a square array of $\Delta(\omega_a)$ centered on the term $-1 + \frac{\Lambda_1 \nu^2 + \Lambda_2}{\omega_a^2}$. This term, which originally was associated with $C_o$ in equation (15), will be referred to hereinafter as the "origin" of the infinite determinant $\Delta(\omega_a)$. The choice of this term as center of $\Delta(\omega_a)$ is purely arbitrary, and it was selected solely for reasons of symmetry.

Thus

$$\Delta(\omega_a) = \lim_{n \to \infty} \Delta_n(\omega_a) \tag{17}$$

The limiting values of the roots of the equation

$$\Delta_n(\omega_a) = 0$$

as $n$ becomes infinite will be the values of the roots of the equation

$$\Delta(\omega_a) = 0$$

The method of calculating the roots of $\Delta(\omega_a) = 0$, by successively calculating the roots of $\Delta_n(\omega_a) = 0$ for larger values of $n$, is entirely too tedious. Instead, the method of reference 3 will be followed. This method involves the calculation of the value of $\Delta(\omega_a)$ for several specified values of $\omega_a$. The roots of $\Delta(\omega_a) = 0$ can then be obtained from a trigonometric equation involving the roots and the calculated values of $\Delta(\omega_a)$.

## Auxiliary Determinants and Recurrence Relations for Calculating $\Delta(\omega_a)$

Before the trigonometric equation is derived, it is convenient to have a systematic numerical procedure for determining the value of $\Delta(\omega_a)$. As $n$ becomes infinite, the terms of $\Delta_n(\omega_a)$ extend to infinity both above and below the origin. By expanding $\Delta_n(\omega_a)$ in terms of the elements of the column containing the origin, it can be expressed in terms of auxiliary determinants that extend to infinity in only one direction. Recurrence relations can then be obtained that give the value of these auxiliary determinants.

The auxiliary determinants are minors of $\Delta_n(\omega_a)$ and are defined as follows:

$C_n(\omega_a)$ determinant of order $3n - 2$ consisting of the terms below and to the right of the origin; that is, determinant having first row and column beginning with term $-1 + \dfrac{K/M}{(\omega_a + \omega)^2}$

$D_n(\omega_a)$ determinant of order $3n - 3$ formed from $C_n(\omega_a)$ by omitting last row and column

$E_n(\omega_a)$ determinant of order $3n - 4$ formed from $D_n(\omega_a)$ by omitting last row and column

$L_n(\omega_a)$ determinant of order $3n - 3$ formed from $C_n(\omega_a)$ by omitting first row and column

$M_n(\omega_a)$ determinant of order $3n - 4$ formed from $L_n(\omega_a)$ by omitting last row and column

$N_n(\omega_a)$ determinant of order $3n - 5$ formed from $M_n(\omega_a)$ by omitting last row and column

The following three determinants will also be needed:

$G_n(\omega_a)$ determinant of order $3n - 4$ formed from $L_n(\omega_a)$ by omitting first row and column

$H_n(\omega_a)$ determinant of order $3n - 5$ formed from $G_n(\omega_a)$ by omitting last row and column

$I_n(\omega_a)$ determinant of order $3n - 6$ formed from $H_n(\omega_a)$ by omitting last row and column

Determinants similar to the foregoing can be formed in the same manner from the upper half of $\Delta_n(\omega_a)$. Denote these determinants by the subscript $-n$ instead of $n$. It is seen, however, that their values can be obtained from the values of the determinants already defined from the lower half of $\Delta_n(\omega_a)$, by merely replacing $\omega_a$ with $-\omega_a$ (for example, $C_{-n}(\omega_a) = C_n(-\omega_a)$).

Expanding $\Delta_n(\omega_a)$ in terms of the elements of the column containing the origin gives

$$\Delta_n(\omega_a) = \left(-1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{\omega_a^2}\right) C_n(\omega_a)\ C_n(-\omega_a) - \Lambda_3\left[\left(\frac{\omega_a - \omega}{\omega_a}\right)^2 I_n(-\omega_a)\ C_n(\omega_a) + \left(\frac{\omega_a + \omega}{\omega_a}\right)^2 I_n(\omega_a)\ C_n(-\omega_a)\right] \tag{18}$$

The auxiliary determinants $C_n(\omega_a)$, $D_n(\omega_a)$, and $E_n(\omega_a)$ satisfy the following recurrence relations (obtained by expanding each in terms of the elements of its last row):

$$\left.\begin{aligned}
C_n(\omega_a) &= \left\{-1 + \frac{K/M}{\left[\omega_a + (2n-1)\omega\right]^2}\right\} D_n(\omega_a) - \Lambda_3\left[\frac{\omega_a + (2n-1)\omega}{\omega_a + (2n-2)\omega}\right]^2 E_n(\omega_a) \\
D_n(\omega_a) &= \left\{-1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{\left[\omega_a + (2n-2)\omega\right]^2}\right\} E_n(\omega_a) - \Lambda_3\left[\frac{\omega_a + (2n-3)\omega}{\omega_a + (2n-2)\omega}\right]^2 C_{n-1}(\omega_a) \\
E_n(\omega_a) &= \left\{-1 + \frac{K/M}{\left[\omega_a + (2n-3)\omega\right]^2}\right\} C_{n-1}(\omega_a) - \frac{(\Delta K/M)^2}{\left[\omega_a + (2n-3)\omega\right]^4} D_{n-1}(\omega_a)
\end{aligned}\right\} \tag{19}$$

The determinants $L_n(\omega_a)$, $M_n(\omega_a)$, and $N_n(\omega_a)$, and also the system $G_n(\omega_a)$, $H_n(\omega_a)$, and $I_n(\omega_a)$ satisfy the same recurrence relations as $C_n(\omega_a)$, $D_n(\omega_a)$, and $E_n(\omega_a)$, respectively.

The values of these nine determinants can be found from the recurrence relations (equations (19)) and the following initial values, obtained directly from equation (16):

$$
\left.
\begin{aligned}
C_1(\omega_a) &= -1 + \frac{K/M}{(\omega_a + \omega)^2} \\
E_2(\omega_a) &= \left[-1 + \frac{K/M}{(\omega_a + \omega)^2}\right]^2 - \frac{(\Delta K/M)^2}{(\omega_a + \omega)^4} \\
N_2(\omega_a) &= -1 + \frac{K/M}{(\omega_a + \omega)^2} \\
M_2(\omega_a) &= \left[1 - \frac{K/M}{(\omega_a + \omega)^2}\right]\left[1 - \frac{\Lambda_1\omega^2 + \Lambda_2}{(\omega_a + 2\omega)^2}\right] - \Lambda_3\left(\frac{\omega_a + \omega}{\omega_a + 2\omega}\right)^2 \\
E_2(\omega_a) &= -1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{(\omega_a + 2\omega)^2} \\
G_2(\omega_a) &= \left[1 - \frac{\Lambda_1\omega^2 + \Lambda_2}{(\omega_a + 2\omega)^2}\right]\left[1 - \frac{K/M}{(\omega_a + 3\omega)^2}\right] - \Lambda_3\left(\frac{\omega_a + 3\omega}{\omega_a + 2\omega}\right)^2
\end{aligned}
\right\} \qquad (20)
$$

By use of the initial conditions (equations (20)), the recurrence relations (equations (19)), and equation (18) the value of $\Delta_n(\omega_a)$ can be calculated. The value of $\Delta(\omega_a)$ will then be the limiting value of $\Delta_n(\omega_a)$ as $n$ becomes infinite.

## The Behavior of $\Delta_n(\omega_a)$ for Large Values of $n$

So far it has been tacitly assumed that the determinant $\Delta(\omega_a)$, as defined in equations (16) and (17), is convergent, and further, that it remains a function of $\omega_a$ in the limit as $n$ becomes infinite; that is, it is not identically equal to zero, independent of the value of $\omega_a$. It will now be shown that the function $\Delta_n(\omega_a)$ does become zero in the limit, independent of $\omega_a$, but that when $\Delta_n(\omega_a)$ is divided by an appropriate function of $n$, a new function $F_n(\omega_a)$ will be obtained which will be convergent and remain an unambiguous function of $\omega_a$ in the limit.

The derivation of the appropriate function by which to divide $\Delta_n(\omega_a)$ evidently depends upon the behavior of $\Delta_n(\omega_a)$ as $n$ becomes very large. As $n$ becomes infinite, the recurrence relations (equations (19)) become

$$C_n = -D_n - \Lambda_3 E_n \tag{21a}$$

$$D_n = -E_n - \Lambda_3 C_{n-1} \tag{21b}$$

$$E_n = -C_{n-1} \tag{21c}$$

with identical equations for $L_n$, $M_n$, and $N_n$ and for $G_n$, $H_n$, and $I_n$. Equations (21) are readily solvable since they constitute a system of difference equations with constant coefficients. They are satisfied by solutions of the form

$$C_n = C_o k^n \tag{22a}$$

where $C_o$ is some arbitrary constant and $k$ is a constant to be determined. From equations (21c) and (21b), respectively,

$$E_n = -C_o k^{n-1} \tag{22b}$$

and

$$D_n = C_o k^{n-1} - \Lambda_3 C_o k^{n-1}$$

$$= C_o\left(1 - \Lambda_3\right)k^{n-1} \tag{22c}$$

Combining equations (22a), (22b), and (22c) with equation (21a) and dividing through by $C_o k^{n-1}$ gives

$$k = 2\Lambda_3 - 1 \tag{23}$$

Thus, by use of equation (18), it is seen that for large values of $n$, $\Delta_n(\omega_a)$ varies as $k^{2n}$. Since by definition $\Lambda_3$ must have a value between 0 and 1/2, $k^2$ must lie between 0 and 1. Thus, in the limit

$$\lim_{n \to \infty} \Delta_n(\omega_a) = 0$$

independent of $\omega_a$.

Consider the function

$$F_n(\omega_a) = \frac{\Delta_n(\omega_a)}{k^{2n}}$$

The equation $F_n(\omega_a) = 0$ will obviously have the same roots for $\omega_a$ as does $\Delta_n(\omega_a) = 0$. The function $F_n(\omega_a)$ has the advantage, however, as is seen from the preceding discussion, of remaining an unambiguous function of $\omega_a$ in the limit as $n$ becomes infinite. Define this limit as

$$\begin{aligned} F(\omega_a) &= \lim_{n \to \infty} F_n(\omega_a) \\ &= \lim_{n \to \infty} \frac{\Delta_n(\omega_a)}{k^{2n}} \end{aligned} \tag{24}$$

The primary problem can now be redefined as the problem of determining the roots, infinite in number and consisting of $\omega_{a_1}$, $\omega_{a_2}$, and $\omega_{a_3}$ and all their harmonics, of the equation

$$F(\omega_a) = 0 \tag{25}$$

## Evaluation of Roots of Equation (25)

The following trigonometric expression for $F(\omega_a)$ will now be derived:

$$F(\omega_a) \equiv \lim_{n \to \infty} \frac{\Delta_n(\omega_a)}{k^{2n}}$$

$$= \frac{1}{k} \frac{\prod_{j=1}^{3} \left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a_j}}{2\omega}\right)\right]}{\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) \cos^4\left(\frac{\pi\omega_a}{2\omega}\right)} \tag{26}$$

The function $F(\omega_a)$ is seen from equation (24) and equations (16) to be periodic of period $2\omega$, to have roots $\pm(\omega_{a_1} \pm 2s\omega)$, $\pm(\omega_{a_2} \pm 2s\omega)$, and $\pm(\omega_{a_3} \pm 2s\omega)$, where $s$ is any integer, to have second-order poles at $\omega_a = \pm 2s\omega$, and to have fourth-order poles at $\omega_a = \pm(2s + 1)\omega$. Liouville's function theorem states that a function of a complex variable (in this case, $\omega_a$) that is analytic everywhere in the complex plane, including the region at infinity, must be a constant. It will be shown that $F(\omega_a)$ is finite at infinity (except if $\omega_a$ proceeds to infinity along the real axis). If the poles along the real axis could be eliminated by forming a suitable function of $F(\omega_a)$, without at the same time introducing new poles, then that function, by Liouville's theorem, must be a constant.

Such a function of $F(\omega_a)$, which is analytic everywhere in the complex plane, is

$$J(\omega_a) = \frac{F(\omega_a)\ \sin^2\left(\frac{\pi\omega_a}{2\omega}\right)\ \cos^4\left(\frac{\pi\omega_a}{2\omega}\right)}{\prod_{j=1}^{3}\left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a_j}}{2\omega}\right)\right]} \tag{27}$$

where $\omega_{a_1}$, $\omega_{a_2}$, and $\omega_{a_3}$ are the three principal values of $\omega_a$. The function $J(\omega_a)$ is therefore a constant. The value of $J(\omega_a)$ found by making $\omega_a$ approach infinity along the imaginary axis is

$$J(\omega_a) = F(\infty)$$

where $F(\infty)$ is the value of $F(\omega_a)$, as $\omega_a$ becomes infinite.

The value of $F(\infty)$ can be found by letting $\omega_a \rightarrow \infty$ in $\Delta_n(\omega_a)$ and then letting $n \rightarrow \infty$. From the form of $\Delta_n(\infty)$ it follows that

$$\Delta_n(\infty) = N_{2n}(\infty) = -C_{2n}(\infty)$$

The recurrence relations defining $C_n(\infty)$ are the same as equations (21). The expressions $D_n$ and $E_n$ may be eliminated from equations (21), which gives

$$C_n = -(1 - 2\Lambda_3)C_{n-1}$$

The initial conditions (equations (20)) reduce to $C_1 = -1$, from which it follows immediately that

$$\Delta_n(\infty) = -\left(1 - 2A_3\right)^{2n-1}$$

$$= k^{2n-1}$$

from equation (23). Therefore

$$F(\infty) \equiv \lim_{n \to \infty} \frac{\Delta_n(\infty)}{k^{2n}} = \lim_{n \to \infty} \frac{k^{2n-1}}{k^{2n}} = \frac{1}{k} \tag{28}$$

Equations (27) and (28) immediately give equation (26).

After equation (26) has been obtained, the problem of determining $\omega_{a_1}$, $\omega_{a_2}$, and $\omega_{a_3}$ may be considered theoretically complete, for equation (26) is really an identity in $\omega_a$. Suppose that $\omega_a$ is assigned any specific value in equation (26) and $F(\omega_a)$ is computed to a certain degree of accuracy. If these computations are made for two more values of $\omega_a$, all different, equation (26) will have yielded three equations in the three unknowns $\omega_{a_1}$, $\omega_{a_2}$, and $\omega_{a_3}$. These equations can then be solved for the principal values of $\omega_a$. Any degree of accuracy may be achieved by carrying out the computations for $F(\omega_a)$ to a sufficiently large value of $n$.

The foregoing procedure can be systematized by rewriting equation (26) as

$$K\left(\frac{\omega_a}{\omega}\right) \equiv \prod_{j=1}^{3} \left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a_j}}{2\omega}\right)\right]$$

$$= kF(\omega_a)\,\sin^2\left(\frac{\pi\omega_a}{2\omega}\right)\,\cos^4\left(\frac{\pi\omega_a}{2\omega}\right) \tag{29}$$

A convenient choice for the three arbitrary values of $\omega_a$ is $\omega_a = 0$, $\omega$, and $\omega/2$.

The explicit definitions of $K(0)$, $K(1)$, and $K(1/2)$ then become

$$\left.\begin{aligned}
K(0) &= -\sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right) \sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right) \sin^2\left(\frac{\pi\omega_{a3}}{2\omega}\right) \\
K(1) &= \left[1 - \sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)\right]\left[1 - \sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right)\right]\left[1 - \sin^2\left(\frac{\pi\omega_{a3}}{2\omega}\right)\right] \\
K(1/2) &= \left[\frac{1}{2} - \sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)\right]\left[\frac{1}{2} - \sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right)\right]\left[\frac{1}{2} - \sin^2\left(\frac{\pi\omega_{a3}}{2\omega}\right)\right]
\end{aligned}\right\} \quad (30)$$

The equations for evaluating $K(0)$, $K(1)$, and $K(1/2)$ are

$$\left.\begin{aligned}
K(0) &= \lim_{\omega_a \to 0} \left[kF(\omega_a) \sin^2\left(\frac{\pi\omega_a}{2\omega}\right) \cos^4\left(\frac{\pi\omega_a}{2\omega}\right)\right] \\
K(1) &= \lim_{\omega_a \to \omega} \left[kF(\omega_a) \sin^2\left(\frac{\pi\omega_a}{2\omega}\right) \cos^4\left(\frac{\pi\omega_a}{2\omega}\right)\right] \\
K(1/2) &= \lim_{\omega_a \to \omega/2} \left[kF(\omega_a) \sin^2\left(\frac{\pi\omega_a}{2\omega}\right) \cos^4\left(\frac{\pi\omega_a}{2\omega}\right)\right]
\end{aligned}\right\} \quad (31)$$

Carrying out the limiting operations indicated in equations (31), and using the auxiliary determinants $C_n(\omega_a)$, and so forth, give

$$
\left.
\begin{aligned}
K(0) &= \lim_{n\to\infty}\left\{\frac{\pi^2\left[\left(\Lambda_1\omega^2+\Lambda_2\right)C_n^{\,2}(0) - 2\Lambda_3\omega^2 L_n(0)C_n(0)\right]}{-4\omega^2(1-2\Lambda_3)^{2n-1}}\right\} \\
K(1) &= \lim_{n\to\infty}\left[\frac{\pi^4\dfrac{K_x}{M}\dfrac{K_y}{M}C_n^{\,2}(-\omega)}{16(1-2\Lambda_3)^{2n-2}}\right] \\
K(1/2) &= \lim_{n\to\infty}\left\{\frac{\left[-1+\dfrac{4(\Lambda_1\omega^2+\Lambda_2)}{\omega^2}\right]C_n\left(\frac{\omega}{2}\right)C_n\left(\frac{-\omega}{2}\right) - \Lambda_3\left[L_n\left(\frac{-\omega}{2}\right)C_n\left(\frac{\omega}{2}\right) + 9L_n\left(\frac{\omega}{2}\right)C_n\left(\frac{-\omega}{2}\right)\right]}{-8(1-2\Lambda_3)^{2n-1}}\right\}
\end{aligned}
\right\}
\tag{32}
$$

where the quantities in brackets are conveniently represented by $K(0)_n$, $K(1)_n$, and $K(1/2)_n$, respectively. The quantities $K(\ )_n$ are used in numerical computations as approximations to the functions $K(\ )$.

The formulas (32) for $K(0)$, $K(1)$, and $K(1/2)$ converge slowly with increasing $n$. The convergence can be speeded up greatly by making use of the concept of convergence factors used in reference 3. A convergence factor for $K(\ )_n$ is a function of $n$ approaching the limit 1 as $n$ becomes infinite, which, when multiplied by $K(\ )_n$, gives an expression which converges rapidly with increasing values of $n$ to the values of $K(\ )$. The details of the derivation of an appropriate convergence factor for $K(0)_n$ will be found in appendix B. Convergence factors for $K(1)_n$ and $K(1/2)_n$ are derived in a similar fashion. The resultant expressions are

$$\left.\begin{aligned}
K(0) &= \lim_{n \to \infty} \left[ \frac{K(0)_n \sin^2 \frac{\pi\sqrt{Q}}{2}}{\frac{\pi^2 Q}{4} \prod_{j=1}^{n-1} \left(1 - \frac{Q}{4j^2}\right)^2} \right] \\
K(1) &= \lim_{n \to \infty} \left[ \frac{K(1)_n \cos^2 \frac{\pi\sqrt{Q}}{2}}{\prod_{j=1}^{n-1} \left(1 - \frac{Q}{(2j-1)^2}\right)^2} \right] \\
K(1/2) &= \lim_{n \to \infty} \left[ \frac{K(1/2)_n \cos \pi\sqrt{Q}}{\prod_{j=1}^{2n-1} \left(1 - \frac{4Q}{(2j-1)^2}\right)} \right]
\end{aligned}\right\} \qquad (33)$$

where

$$Q = \frac{2\frac{K}{M}(1 - \Lambda_3) + \Lambda_1\omega^2 + \Lambda_2 + 2\Lambda_3\omega^2}{\omega^2(1 - 2\Lambda_3)}$$

For a given value of $n$, the quantities in brackets are found to be better approximations to the respective values of $K(\ )$ than $K(\ )_n$ alone.

The method of obtaining the values of $\omega_a$ may be summarized as follows: by use of the initial conditions (equations (20)) and the recurrence relations (equations (19)), the values of the determinants $C_n(0)$, $L_n(0)$, $G_n(\pm\omega)$, $C_n(\omega/2)$, $C_n(-\omega/2)$, $L_n(\omega/2)$, and $L_n(-\omega/2)$ can be computed for increasing values of $n$. With the substitution of these values into equations (33), and with the use of equations (32), approximate values of $K(0)$, $K(1)$, and $K(1/2)$ can be computed.

The process appears to be rapidly convergent with $n$, especially for large values of $\omega/\sqrt{K/M}$. The values of $\omega_{a_1}$, $\omega_{a_2}$, and $\omega_{a_3}$ can then be found from equations (31), the definitions of $K(0)$, $K(1)$, and $K(1/2)$.

## Conditions for Stability

From equation (13) the condition for stability of the system is seen to be that all three values of $\omega_a$ must be real numbers. If any one of them is complex or pure imaginary, then one of the terms in the solution (equations (11)) will increase indefinitely with the time, the motion therefore being unstable. This condition implies that the expressions $\sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)$, $\sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right)$, and $\sin^2\left(\frac{\pi\omega_{a3}}{2\omega}\right)$ all are real positive numbers less than or equal to 1. The conditions for stability can be expressed directly in terms of $K(0)$, $K(1)$, and $K(1/2)$ by means of their definitions (equations (30)). The three equations (30) are formally equivalent to a single cubic equation

$$x^3 + bx^2 + cx + d = 0$$

the roots $x_1$, $x_2$, and $x_3$ of which are $\sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)$ and so forth, and the coefficients $b$, $c$, and $d$ of which are functions of $K(0)$, $K(1)$, and $K(1/2)$ where

$$2b = 4K(0) + 4K(1) - 8K\left(\frac{1}{2}\right) - 3$$

$$2c = -6K(0) - 2K(1) + 8K\left(\frac{1}{2}\right) + 1$$

$$d = K(0)$$

After some manipulations involving the Descartes rule of signs, the necessary and sufficient conditions for stability are found to be

$$
\begin{aligned}
0 &\leqq -K(0) \leqq 1 \\
0 &\leqq K(1) \leqq 1 \\
-1 &\leqq 8K(1/2) \leqq 1 \qquad (34) \\
\Delta &= 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^3 \geqq 0
\end{aligned}
$$

The quantity $\Delta$ is the discriminant of the cubic equation.

## SPECIAL CASES OF GENERAL THEORY

Three special cases of the general theory are of interest. These cases are the cases for which one of the principal stiffnesses $K_y$ is respectively zero, equal, or infinite in magnitude in comparison with the second principal stiffness $K_x$.

### Case of $K_y = K_x$

The case of $K_y = K_x$ has been treated in reference 2. If $K_y = K_x$, the equations (8) to (10) reduce to the equations of reference 2. In this special case, the motion of the rotor system becomes simple harmonic, since all the coefficients $A_l$, $B_l$, and $C_l$ in equations (12) are identically zero except $A_o$, $B_o$, and $C_o$.

### Case of $K_y = 0$

The special limiting case of $K_y = 0$ is of interest in the case of a pylon of which the stiffness is negligible along one principal direction with interest centered on the frequencies involving the other principal stiffness. In the case of $K_y = 0$, the function $K(1)$ as given by equations (32) becomes identically zero. This result is also evident from the original definition of $K(1)$ as given in equation (28), because one of the values of $\omega_a$, say $\omega_{a_3}$, is of necessity equal to $\pm\omega$. (It will be recalled that $\omega_a$ is the frequency as measured in rotating coordinates. In fixed coordinates it would be zero.)

It is possible to give much simpler stability criterions for this case because there are only two K-functions, $K(0)$ and $K(1/2)$, and two values of $\omega_a$, $\omega_{a_1}$ and $\omega_{a_2}$, to be determined. The new K-functions may be defined as follows:

$$\left.\begin{aligned} K_1 &= \sin^2\left(\frac{\pi\omega_{a_1}}{2\omega}\right)\sin^2\left(\frac{\pi\omega_{a_2}}{2\omega}\right) = -K(0) \\ K_2 &= \cos^2\left(\frac{\pi\omega_{a_1}}{2\omega}\right)\cos^2\left(\frac{\pi\omega_{a_2}}{2\omega}\right) = \frac{1 + 2K(0) - 8K\left(\frac{1}{2}\right)}{2} \end{aligned}\right\} \tag{35}$$

In terms of $K_1$ and $K_2$, the criterions for stability become

$$0 \leqq K_1 \leqq 1 \tag{36a}$$

$$0 \leqq K_2 \leqq 1 \tag{36b}$$

$$\sqrt{K_1} + \sqrt{K_2} \leqq 1 \tag{36c}$$

Given the values of $K_1$ and $K_2$, the values of $\omega_{a_1}$ and $\omega_{a_2}$ can be determined from equations (35). A graph of the relation in equation (35) is given in figure 2 by means of which the real values of $\omega_{a_1}$ and $\omega_{a_2}$ can be read off directly once $K_1$ and $K_2$ are known.

A graph showing the variation of $K_1$ and $K_2$ with $\omega/\sqrt{K_x/M}$, for the typical parameters $\Lambda_1 = 0.1$, $\Lambda_2 = 0$, $\Lambda_3 = 0.1$, and $K_y = 0$, is shown in figure 3. By use of figure 2 the values of $\omega_{a_1}$

and $\omega_{a_2}$ can be obtained. These values are shown in figure 4 plotted against $\omega/\sqrt{K_x/M}$. Calculations are carried down to $\frac{\omega}{\sqrt{K_x/M}} = 0.5$. The general behavior below this speed is discussed in the section entitled "General Behavior of Rotor System as a Function of Rotor Speed."

## Case of $K_y = \infty$

The formulas for the limiting case of $K_y = \infty$ cannot be obtained conveniently from the general theory. Instead of carrying out the limiting process, it appears preferable to begin by treating the problem as one of only three degrees of freedom (two hinge-deflection coordinates and one hub-position coordinate $x$), and by developing the theory along lines similar to those used for the general treatment. In this way a system of two simultaneous equations with periodic coefficients is obtained, with the variables $\theta_1$ and $x$. These equations are solved in a manner similar to that for the general case, the treatment being simpler, however, since the solution has only two principal values of $\omega_a$.

The details of the solution of the equations of motion, together with the final formulas for the K-functions, including convergence factors, are given in appendix C. It is found that the same $K_1$ and $K_2$ occur as for $K_y = 0$. The criterions for stability are exactly the same as those for $K_y = 0$, the conditions of equations (36). Figure 2 can also be used to determine the values of $\omega_a$ from the values of $K_1$ and $K_2$.

A graph giving the variation of $K_1$ and $K_2$ with $\omega/\sqrt{K_x/M}$ for the parameters $\Lambda_1 = 0.1$, $\Lambda_2 = 0$, $\Lambda_3 = 0.1$, and $K_y = \infty$ is shown in figure 5. In figure 6 the values of $\omega_{a_1}/\sqrt{K_x/M}$ and $\omega_{a_2}/\sqrt{K_x/M}$ are shown plotted against $\omega/\sqrt{K_x/M}$.

# DISCUSSION OF RESULTS

## Types of Instability

Instability may occur as a result of the violation of any one of the stability criterions of equations (34). Violation of each condition is associated with a different type of instability, which would show up differently in the motion of the rotor system. Experience with computations indicates, however, that the criterions of most practical importance for helicopters are

$$\Delta \geqq 0$$

$$-K(0) \geqq 0$$

$$K(1) \geqq 0$$

Similarly, the important criterions in the limiting cases of $K_y = 0$ and $K_y = \infty$ are

$$\sqrt{K_1} + \sqrt{K_2} \leqq 1$$

$$K_1 \geqq 0$$

$$K_2 \geqq 0$$

If the condition $\Delta \geqq 0$ (or $\sqrt{K_1} + \sqrt{K_2} \leqq 1$) is violated, the other conditions being satisfied, then $\omega_{a_1}$ and $\omega_{a_2}$ will be complex conjugates, and the rotor system will execute self-excited vibrations at frequencies, in general, incommensurate with the rotor speed. (Higher harmonics will also be present.) This type of instability will be referred to hereinafter as a "self-excited vibration."

If the stability condition $-K(0) \geqq 0$ (or $K_1 \geqq 0$) alone is violated, then one of the values of $\omega_a$ will be a pure imaginary number. Physically, the rotor system will execute self-excited vibrations having a basic frequency, as seen in rotating coordinates, of zero. This behavior is similar to the ordinary critical-speed behavior of a shaft. Frequencies at higher harmonics $2n\omega$ will also be present. This type of instability will be referred to as a "self-excited whirling."

The third stability condition $K(1) \geqq 0$ cannot be violated since $K(1)$ as given by equation (32) cannot be negative. However, $K(1)$ can be exactly equal to zero. (A similar statement applies to $K_2$.)

At such a point, where the rotor system is on the border line between stability and instability, one of the values of $\omega_a$ will be equal to $\pm\omega$. In fixed coordinates this result means that the rotor system will have a natural frequency equal to zero. The rotor system will, therefore, be in resonance with a steady force - a force of zero frequency. The amplitude of the zero-frequency term for the hub motion in such a situation can be shown to be zero, but the blades will oscillate. Also, higher harmonic terms, notably the term of frequency $2\omega$ (in nonrotating coordinates), will show up in the hub motion. This type of vibration, which is a resonance phenomenon and not a self-excited vibration, will be called a "steady-force resonance" vibration.

Each of the vibrations described - self-excited vibrations, self-excited whirling, and a steady-force resonance vibration - appeared in the discussion of the two-blade rotor on equal supports (reference 2); however, there the motions were simple harmonic, no higher harmonics being present.

## General Behavior of Rotor System as a Function of Rotor Speed

The approximate location of the instability regions can easily be found by examining the limiting case of $\Lambda_3 = 0$, that is, the case of zero coupling between the blade and hub motions. For simplicity, the discussion is also restricted to the case of free hinges $(\Lambda_2 = 0)$ and $K_y = \infty$. The $K_1$ and $K_2$ functions become

$$\left.\begin{aligned} K_1 &= \sin^2\left(\frac{\pi}{2}\sqrt{\Lambda_1}\right) \cos^2\left(\frac{\pi}{2\omega}\sqrt{\frac{K_x}{M}}\right) \\ K_2 &= \cos^2\left(\frac{\pi}{2}\sqrt{\Lambda_1}\right) \sin^2\left(\frac{\pi}{2\omega}\sqrt{\frac{K_x}{M}}\right) \end{aligned}\right\} \qquad (37)$$

Eliminating the rotor speed $\omega$ from equations (37) gives

$$\frac{K_1}{\sin^2\left(\frac{\pi}{2}\sqrt{\Lambda_1}\right)} + \frac{K_2}{\cos^2\left(\frac{\pi}{2}\sqrt{\Lambda_1}\right)} = 1$$

Considered as an equation in the variables $K_1$ and $K_2$ this equation represents a straight-line segment (one of the lines in fig. 2) terminated by the $K_1$ and $K_2$ axes. The segment can be shown to be tangent to the curve $\sqrt{K_1} + \sqrt{K_2} = 1$. As $\omega$ decreases, the representative point moves up and down the line segment, performing an infinite number of such oscillations as $\omega$ approaches zero. Whenever $K_1 = 0$, the point is at a self-excited-whirling speed. The corresponding speed is

$$\omega = \frac{\sqrt{K_x/M}}{2s + 1}$$

where s represents any positive integer. Thus a self-excited whirling will occur when the rotor speed is approximately equal to 1, 1/3, 1/5, 1/7, and so forth of the natural frequency of the hub $\sqrt{K_x/M}$. Similarly it can be shown that there will be a steady-force resonance vibration whenever the rotor speed is approximately equal to 1/2, 1/4, 1/6, and so forth of the hub natural frequency $\sqrt{K_x/M}$. Finally self-excited vibrations will occur at rotor speeds approximately equal to

$$\omega = \frac{\sqrt{K_x/M}}{2s + 1 - \sqrt{\Lambda_1}}$$

Figure 7 shows the general pattern of response frequency plotted against rotor speed for a small value of the mass-ratio parameter $\Lambda_3$. The variable $\omega_a/\omega$ rather than $\omega_a\sqrt{K_x/M}$ has been plotted as ordinate to avoid crowding of the lines. Along the horizontal parts of the curves, blade motion predominates over pylon motion. Pylon motion predominates along the slanting parts.

Although the foregoing discussion was developed for the case of $K_y = \infty$, it is believed to apply equally well to the case of $K_y = 0$ and also to the general case of $K_y \neq K_x$ if the rotor hub is considered to have two natural frequencies $\sqrt{K_x/M}$ and $\sqrt{K_y/M}$, each frequency having associated with it an infinite set of instability ranges located at approximately the speeds given.

## Comparison of Results for Different Values of $K_y/K_x$

Figures 4 and 6 give the principal values of $\omega_a/\sqrt{K_x/M}$ plotted against the rotor speed $\omega/\sqrt{K_x/M}$ for $K_y = 0$ and $K_y = \infty$, respectively, both calculated for the same set of parameters $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$. The calculations have been carried down to $\frac{\omega}{\sqrt{K_x/M}} = 0.4$. The similarity between the two curves is striking. So far as the calculations have been carried, each system shows the presence of one self-excited-vibration instability range, one self-excited-whirling instability range, and one steady-force resonance speed A. If the calculations were carried to lower values of $\omega$, further instability ranges and steady-force resonance speeds would appear.

For comparison, the response frequencies of a two-blade rotor on equal supports $(K_y = K_x)$ for the same set of parameters is shown in figure 8. The frequencies were calculated from the formula in reference 2. Down to $\frac{\omega}{\sqrt{K_x/M}} = 0.5$, this chart is very similar to figures 4 and 6. In addition it shows one range of rotational speed at which self-excited-vibration instability occurs, one range of rotational speed at which self-excited-whirling instability occurs, and one range of rotational speed at which a steady-force resonance speed occurs. Figure 8 differs principally from the figures for $K_y \neq K_x$ in that it shows no further instability ranges at low values of $\omega$.

In references 1 and 2 charts are presented giving the location of the self-excited-vibration instability range for various values of the parameters $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$. A similar chart for the case of a two-blade rotor with $K_y = \infty$ is given in figure 9. In using the chart, a straight line is drawn representing the variation of $\frac{\omega^2}{K_x/M}$ with the function $\frac{\Lambda_1\omega^2 + \Lambda_2}{K_x/M}$. The intersections of this line with the appropriate $\Lambda_3$ curves give the beginning and end points of the instability range. The dashed line in figure 9 illustrates the method for the parameters of figure 6.

Some observations concerning the relative location and extent of the various instability ranges in figures 4, 6, and 8 appear to be applicable to a wide range of values of the parameters $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$. Thus the self-excited-vibration instability range in the case of $K_y = K_x$ (fig. 8), is wider (and hence the vibration probably more severe) than the corresponding ranges in the cases of $K_y = 0$ and $K_y = \infty$. (See figs. 4 and 6.) Also, this instability range occurs at lower rotor speeds in the case of $K_y = \infty$ than it does in the cases of $K_y = K_x$ and $K_y = 0$. The self-excited-whirling instability range is considerably narrower for $K_y = \infty$ than it is for the $K_y = K_x$ case, and it is still narrower in the $K_y = 0$ case.

In the general case of $K_y \neq K_x$ the location and extent of the instability ranges can be found fairly accurately by considering the problem as the superposition of two problems, one of finding the significant rotor speeds referred to $\sqrt{K_x/M}$ as reference frequency with $K_y$ assumed infinite and the other of finding the significant rotor speeds referred to $\sqrt{K_y/M}$ as reference frequency with $K_x$ assumed zero. With the foregoing discussion as a guide, sufficiently accurate design information can be obtained without extensive calculations for each value of $K_y/K_x$ encountered in practice.

### Effect of Damping

Although the effect of damping has not been examined mathematically, because complications would be introduced in the analysis, several inferences from the damping investigations in references 1 and 2 can probably be safely applied to the rotor-system studies in the present paper. The numerous instability ranges occurring at low rotor speeds, which are very narrow and represent a mild type of instability, are probably completely eliminated by the presence of a slight amount of damping in the rotor system. The primary self-excited-vibration instability range can probably be narrowed and eliminated by introducing sufficient damping into both the rotor supports and the blade hinges.

## APPLICATION TO DUAL ROTORS

It is easily shown that the analysis for the case of $K_y = \infty$ applies also to the case of a counterrotating rotor system consisting of two equal two-blade rotors revolving at equal speeds and acting equally upon the same flexible member. The rotors may be on the same shaft or on different shafts so long as the nonrotating flexible member is the same for both rotors. The supports, moreover, may have unequal stiffness in the X- and in the Y-directions, provided that the undeflected blade positions make equal angles with a principal stiffness axis.

The proof consists in showing that the energy expressions for the dual-rotating system can be separated into two independent sets of terms, each of which is of the same form as for a single rotor with $K_y = \infty$. The resulting equations of motion will thus also be the same.

The separation is accomplished by introducing new variables

$$\xi_1 = \frac{1}{2}\left(\theta_{1_{pos}} - \theta_{1_{neg}}\right)$$

$$\xi_o = \frac{1}{2}\left(\theta_{o_{pos}} - \theta_{o_{neg}}\right)$$

and

$$\eta_1 = \frac{1}{2}\left(\theta_{1_{pos}} + \theta_{1_{neg}}\right)$$

$$\eta_o = \frac{1}{2}\left(\theta_{o_{pos}} + \theta_{o_{neg}}\right)$$

where the subscripts pos and neg refer to the $\theta$'s defined for the rotor turning in the positive direction and for the rotor turning in the opposite direction, respectively. The energy expressions become

$$\left.\begin{aligned}
T = \frac{1}{2}M\dot{x}^2 + 2m_b\Bigg[&-2\dot{x}\left(\dot{\xi}_1 \sin \omega t + \omega\xi_1 \cos \omega t\right) \\
&+ \left(1 + \frac{r^2}{b^2}\right)\left(\dot{\xi}_1^{\,2} + \dot{\xi}_o^{\,2}\right) - \frac{a}{b}\omega^2\left(\xi_1^{\,2} + \xi_o^{\,2}\right)\Bigg] \\
+ \frac{1}{2}M\dot{y}^2 + 2m_b\Bigg[&2\dot{y}\left(\dot{\eta}_1 \cos \omega t - \omega\eta_1 \sin \omega t\right) \\
&+ \left(1 + \frac{r^2}{b^2}\right)\left(\dot{\eta}_1^{\,2} + \dot{\eta}_o^{\,2}\right) - \frac{a}{b}\omega^2\left(\eta_1^{\,2} + \eta_o^{\,2}\right)\Bigg] \\
V = \frac{2K_\beta}{b^2}&\left(\xi_1^{\,2} + \xi_o^{\,2} + \eta_1^{\,2} + \eta_o^{\,2}\right) + \frac{1}{2}K_x x^2 + \frac{1}{2}K_y y^2
\end{aligned}\right\} \quad (33)$$

where $M$ is the total mass of the system $(M = m + 4m_b)$.

From equations (38) it is seen that $\xi$ is coupled only with $x$ and $\eta$ is coupled only with $y$. Equations (38) yield equations of motion of the same form as equations (C2) and (C3).

The stability properties for the dual-rotating case are thus exactly the same as in the case of $K_y = \infty$ for the single-rotating two-blade rotor. In particular, figure 9 can be used to find the location of the primary self-excited-vibration instability range. The value of $\Lambda_3$ for the dual-rotating rotor is defined as $\Lambda_3 = \frac{2m_b}{(m + 4m_b)(1 + r^2/b^2)}$ rather than $\Lambda_3 = \frac{m_b}{(m + 2m_b)(1 + r^2/b^2)}$ as for the single-rotating rotor. All other parameters are the same for both cases.

The quantity $\xi_1 \sin \omega t$ can be interpreted physically as the x-component of the displacement of the center of gravity of the blades due to hinge deflections. The quantity $\eta_1 \cos \omega t$ is the corresponding y-component. The separation of the variables means, physically, that the motion of the system can be separated into two independent modes, each of which involves linear motion of the supports along one of the principal stiffness axes.

Similarly, the stability of counterrotating rotor systems of six or more equal blades can be determined from the results of reference 1 with $K_y = \infty$.

## CONCLUSIONS

The following conclusions are indicated by the results of an investigation of the problem of vibration of a two-blade helicopter rotor on supports that have different stiffnesses along the two principal stiffness axes:

1. Many speed ranges are found in which self-excited oscillations can occur. These oscillations are of two types - self-excited vibration and self-excited whirling. There are also many speeds at which steady-force-resonance vibration may occur.

2. The self-excited vibration, self-excited whirling, and steady-force resonance speeds of highest rotor speed for each support natural frequency are recognized as corresponding to those of a two-blade rotor on equal supports, but changed somewhat in position and extent.

3. Mild self-excited-whirling speed ranges exist at rotor speeds approximately 1/3, 1/5, 1/7, and so forth of each support natural frequency. Steady-force resonance speeds exist at approximately 1/2, 1/4, 1/6, and so forth of each support frequency. Self-excited vibrations also occur at certain low rotor speeds. All these mild instability ranges are probably eliminated by the presence of moderate amounts of damping in the system.

4. A familiarity with typical results of limiting cases of the support-spring constants $K_y = \infty$, $K_y = K_x$, and $K_y = 0$ should enable a designer to avoid extensive calculations of cases of unequal support stiffness. In the general case of unequal support stiffness, the location and extent of the instability ranges can be found fairly accurately by considering the problem as the superposition of two problems, one of finding significant rotor speeds referred to one support frequency $\sqrt{K_x/M}$ as reference frequency with $K_y$ assumed infinite and the other of finding the significant rotor speeds referred to the other support frequency $\sqrt{K_y/M}$ as reference frequency with $K_x$ assumed zero.

5. The analysis of a four-blade counterrotating rotor system in which the rotors cross along the principal stiffness axes of the rotor supports leads to the same equations as those considered for the special case of $K_y = \infty$, and the stability properties are given by the investigation of that special case.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., July 22, 1946

# APPENDIX A

## SYMBOLS

| | |
|---|---|
| $a$ | radial position of vertical hinge |
| $a_{00}$, $a_{01}$, and so forth | elements of determinant defined in equation (16) |
| $A_l$, $B_l$, $C_l$ | Fourier coefficients in equation (12) |
| $\overline{A}_l$, $\overline{B}_l$, $\overline{C}_l$ | complex conjugates of $A_l$, $B_l$, and $C_l$, respectively |
| $b$ | distance from vertical hinge to center of mass of blade |
| $C_n(\omega_a)$, $D_n(\omega_a)$, and so forth | minors of the determinate $\Delta_n(\omega_a)$ |
| $D$ | time-derivative operator $\left(\frac{d}{dt}\right)$ |
| $F(\omega_a) = \lim\limits_{n \to \infty} F_n(\omega_a)$ | |
| $F_n(\omega_a) = \dfrac{\Delta_n(\omega_a)}{k^{2n}}$ | |
| $j$, $n$, $s$ | integers |
| $J(\omega_a)$ | function of $F(\omega_a)$ defined by equation (27) |
| $k = 2\Lambda_3 - 1$ | |
| $K(\omega_a/\omega)$ | function of $\omega_a/\omega$ defined in equation (29) |
| $K(0)$, $K(1)$, $K(1/2)$ | functions defined by the relations (30) |
| $K_1$, $K_2$ | functions defined by the relations (35) |
| $K_x$, $K_y$ | spring constants of the rotor supports along the X- and Y-directions, respectively |

| | |
|---|---|
| $K = \frac{K_y + K_x}{2}$ | average stiffness of rotor supports |
| $\Delta K = \frac{K_y - K_x}{2}$ | |
| $K_\beta$ | spring constant of blade self-centering spring |
| $m$ | effective mass of rotor supports |
| $m_b$ | mass of rotor blade |
| $M$ | total mass of two-blade rotor system $(m + 2m_b)$ |
| $P(t), Q(t), R(t)$ | periodic functions defined in equation (11) |
| $\overline{P}(t), \overline{Q}(t), \overline{R}(t)$ | complex conjugates of $P(t), Q(t)$, and $R(t)$, respectively |
| $Q$ | constant defined in equations (33) |
| $r$ | radius of gyration of blade about center of mass |
| $t$ | time |
| $T$ | kinetic energy |
| $V$ | potential energy |
| $x, y$ | deflection of rotor hub measured in $X,Y$-coordinate system |
| $X, Y$ | fixed rectangular coordinate axes taken parallel to the principal stiffness directions of the rotor hub |
| $z$ | complex position coordinate of the rotor hub in rectangular coordinate system rotating with angular velocity $(x_r + iy_r)$ |
| $\overline{z}$ | complex conjugate of $z$ $(x_r - iy_r)$ |
| $z_f$ | complex position vector in $X,Y$- (nonrotating) coordinate system $(x + iy)$ |

$\beta_1$, $\beta_2$ — angular hinge deflections of rotor blades, respectively

$\Delta$ — discriminant of cubic equation

$$x^3 + bx^2 + cx + d = 0$$

$\Delta(\omega_a)$ — determinant of infinite order defined by equation (16)

$\Delta_n(\omega_a)$ — determinant of order $6n - 3$ formed from $\Delta(\omega_a)$

$$\theta_0 = \frac{b}{2}(\beta_1 + \beta_2)$$

$$\theta_1 = \frac{b}{2}(\beta_1 - \beta_2)$$

$$\Lambda_1 = \frac{a}{b\left(1 + \frac{r^2}{b^2}\right)}$$

$$\Lambda_2 = \frac{K_\beta}{m_b b^2\left(1 + \frac{r^2}{b^2}\right)}$$

$$\Lambda_3 = \frac{\mu}{2\left(1 + \frac{r^2}{b^2}\right)}$$

$\xi_1$, $\xi_0$, $\eta_1$, $\eta_0$ — blade variables for counterrotating rotor

$\mu$ — mass ratio $\left(\frac{2m_b}{M}\right)$

$\omega$ — constant angular velocity of rotor

$\omega_a$ — characteristic exponent or natural frequency of rotor system as viewed in coordinates rotating at angular velocity $\omega$

$\omega_{a_1}$, $\omega_{a_2}$, $\omega_{a_3}$ — principal values of $\omega_a$

## APPENDIX B

## DERIVATION OF THE CONVERGENCE FACTOR FOR K(0)

A convergence factor for $K(0)$ is found by finding a simpler function $G_n$ that changes with $n$ in nearly the same way as $K(0)_n$. Then, if $G$ denotes $\lim_{n \to \infty} G_n$, the expression

$$\frac{K(0)_n G}{G_n}$$

for a given value of $n$ is a better approximation to $K(0)$ than is $K(0)_n$ alone. A suitable form for $G_n$ is found from a study of the behavior of $K(0)_n$ for large values of $n$.

The behavior of $K(0)_n$ is studied by first observing the behavior of $C_n(0)$ and $L_n(0)$ for large values of $n$ and then inferring the behavior of $K(0)_n$ from equation (32). In the discussion of equations (22) it was shown that, as $n$ becomes infinite, the ratio $C_{n+1}/C_n$ approaches the value $k$ (equation (22a)). A closer approximation to the value of this ratio can be written as

$$\frac{C_{n+1}(0)}{C_n(0)} = k\left(1 - \frac{P}{n} - \frac{Q}{4n^2}\right) \tag{B1}$$

where $P$ and $Q$ are constants to be determined.

Equations (19) become, for $\omega_a = 0$,

$$\left.\begin{aligned}
C_{n+1}(0) &= \left[-1 + \frac{K/M}{(2n+1)^2\omega^2}\right] D_n(0) - \Lambda_3\left(\frac{2n+1}{2n}\right)^2 E_n(0) \\
D_{n+1}(0) &= \left[-1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{(2n)^2\omega^2}\right] E_n(0) - \Lambda_3\left(\frac{2n-1}{2n}\right)^2 C_n(0) \\
E_{n+1}(0) &= \left[-1 + \frac{K/M}{(2n-1)^2\omega^2}\right] C_n(0)
\end{aligned}\right\} \qquad \text{(B2)}$$

where the second term in the equation for $E_n(\omega_a)$ in equations (19) has been neglected as being of higher order in powers of $1/n$ than the terms retained. Eliminating $E(0)$ and $D(0)$ from the equations (B2) results in

$$\frac{C_{n+1}(0)}{C_n(0)} = \left[-1 + \frac{K/M}{(2n+1)^2\omega^2}\right]\left[-1 + \frac{\Lambda_1\omega^2 + \Lambda_2}{(2n)^2\omega^2}\right]\left[-1 + \frac{K/M}{(2n-1)^2\omega^2}\right]$$

$$- \Lambda_3\left\{\left(\frac{2n-1}{2n}\right)^2\left[-1 + \frac{K/M}{(2n+1)^2\omega^2}\right] + \left(\frac{2n+1}{2n}\right)^2\left[-1 + \frac{K/M}{(2n-1)^2\omega^2}\right]\right\}$$

Upon expanding this expression into powers of $1/n$ and retaining terms up to and including those in $1/n^2$

$$\frac{C_{n+1}(0)}{C_n(0)} = -1 + 2\Lambda_3 - \frac{1}{(2n)^2}\left[\frac{2\frac{K}{M} + \Lambda_1\omega^2 + \Lambda_2}{\omega^2} + 2\Lambda_3\left(\frac{K/M}{\omega^2} - 1\right)\right] \qquad \text{(B3)}$$

Comparing equation (B3) with equation (B1) results in

$$P = 0$$

$$Q = \frac{2\frac{K}{M}(1 - \Lambda_3) + \Lambda_1\omega^2 + \Lambda_2 + 2\Lambda_3\omega^2}{\omega^2(1 - 2\Lambda_3)}$$

Similarly

$$\frac{L_{n+1}(0)}{L_n(0)} = k\left[1 - \frac{Q}{(2n)^2}\right]$$

Therefore, from equations (32)

$$\frac{K(0)_{n+1}}{K(0)_n} = \left(1 - \frac{Q}{4n^2}\right)^2$$

Hence, an approximate value of $K(0)$ can be obtained from

$$\frac{K(0)}{K(0)_n} = \frac{K(0)_{n+1}}{K(0)_n}\;\frac{K(0)_{n+2}}{K(0)_{n+1}}\;\frac{K(0)_{n+3}}{K(0)_{n+2}}\;\cdot\;\cdot\;\cdot$$

$$\approx \prod_{j=n}^{\infty}\left(1 - \frac{Q}{4j^2}\right)^2 = \frac{\sin^2\left(\frac{\pi\sqrt{Q}}{2}\right)}{\frac{\pi^2 Q}{4}\prod_{j=1}^{n-1}\left(1 - \frac{Q}{4j^2}\right)^2} \tag{B4}$$

The right-hand side of equation (B4), which is seen to be of the form $G/G_n$, is a convergence factor. This convergence factor is the one used in equation (33).

## APPENDIX C

### MATHEMATICAL ANALYSIS FOR THE CASE OF $K_y = \infty$

In terms of the variables $\theta_0$, $\theta_1$, and $x$ the expressions for the kinetic and potential energy are

$$T = \frac{M}{2}\dot{x}^2 + m_b\left[\left(1 + \frac{r^2}{b^2}\right)\left(\dot{\theta}_0^{\,2} + \dot{\theta}_1^{\,2}\right) - \omega^2\frac{a}{b}\left(\theta_0^{\,2} + \theta_1^{\,2}\right) - 2\dot{x}\left(\dot{\theta}_1 \sin \omega t + \omega\theta_1 \cos \omega t\right)\right]$$

$$V = \frac{K_x}{2}x^2 + \frac{K_\beta}{b^2}\left(\theta_0^{\,2} + \theta_1^{\,2}\right)$$

The three equations of motion are

$$\ddot{\theta}_0 + \left(\omega^2\Lambda_1 + \Lambda_2\right)\theta_0 = 0 \qquad \text{(C1)}$$

$$\ddot{x} + \frac{K_x}{M}x - \mu\frac{d^2}{dt^2}\left(\theta_1 \sin \omega t\right) = 0 \qquad \text{(C2)}$$

$$-\frac{1}{\left(1 + \frac{r^2}{b^2}\right)}\ddot{x}\sin \omega t + \ddot{\theta}_1 + \left(\omega^2\Lambda_1 + \Lambda_2\right)\theta_1 = 0 \qquad \text{(C3)}$$

Equation (C1) is identical with equation (7). Equations (C2) and (C3) constitute a system of two linear second-order differential equations with periodic coefficients.

Equations (C2) and (C3) are satisfied by solutions of the form

$$\left.\begin{aligned} x &= \sum_{l=-\infty}^{\infty} A_l e^{i(\omega_a + l\omega)t} \\ \theta_1 &= \sum_{l=-\infty}^{\infty} B_{l+1} e^{i\left[\omega_a + (l+1)\omega\right]t} \end{aligned}\right\} \tag{C4}$$

where $l$ takes on all odd integral values and $A_l$ and $B_{l+1}$ are constants to be determined along with the two principal values of $\omega_a$ $\left(\omega_{a_1}\right.$ and $\left.\omega_{a_2}\right)$. The constants $A_l$ and $B_{l+1}$ in equations (C4) are, of course, different from those in (12).

Combining equations (C4) with equations (C2) and (C3) and setting the coefficients of the various exponential time factors equal to zero

$$\left[-1 + \frac{K_x/M}{(\omega_a + l\omega)^2}\right] A_l - \frac{\mu}{2i}\left(-B_{l-1} + B_{l+1}\right) = 0$$

$$-\frac{1}{2i\left(1 + \frac{r^2}{b^2}\right)}\left\{-\left[\frac{\omega_a + (l-2)\omega}{\omega_a + (l-1)\omega}\right] A_{l-2} + \left[\frac{\omega_a + l\omega}{\omega_a + (l-1)\omega}\right]^2 A_l\right\}$$

$$+ \left\{-1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{\left[\omega_a + (l-1)\omega\right]^2}\right\} B_{l-1} = 0$$

The determinantal equation is then equal to

$$
\begin{vmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_{-3,-3} & a_{-3,-2} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & a_{-2,-3} & a_{-2,-2} & a_{-2,-1} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & a_{-1,-2} & a_{-1,-1} & a_{-1,0} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & a_{0,-1} & a_{0,0} & a_{0,1} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & a_{1,0} & a_{1,1} & a_{1,2} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & a_{3,2} & a_{3,3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{vmatrix} = \Delta(\omega_a) = 0
$$

where

$$a_{-3,-3} = -1 + \frac{K_x/M}{(\omega_a - 3\omega)^2} \qquad a_{0,0} = -1 + \frac{\Lambda_1 \omega^2 + \Lambda_2}{{\omega_a}^2}$$

$$a_{-3,-2} = \frac{\Lambda_3}{2} \qquad a_{0,1} = \left(\frac{\omega_a + \omega}{\omega_a}\right)^2$$

$$a_{-2,-3} = \left(\frac{\omega_a - 3\omega}{\omega_a - 2\omega}\right)^2 \qquad a_{1,0} = \frac{\Lambda_3}{2}$$

$$a_{-2,-2} = -1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{(\omega_a - 2\omega)^2} \qquad a_{1,1} = -1 + \frac{K_x/M}{(\omega_a + \omega)^2}$$

$$a_{-2,-1} = \left(\frac{\omega_a - \omega}{\omega_a - 2\omega}\right)^2 \qquad a_{1,2} = \frac{\Lambda_3}{2}$$

$$a_{-1,-2} = \frac{\Lambda_3}{2} \qquad a_{2,1} = \left(\frac{\omega_a + \omega}{\omega_a + 2\omega}\right)^2$$

$$a_{-1,-1} = -1 + \frac{K_x/M}{(\omega_a - \omega)^2} \qquad a_{22} = -1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{(\omega_a + 2\omega)^2}$$

$$a_{-10} = \frac{\Lambda_3}{2} \qquad a_{23} = \frac{\omega_a + 3\omega}{(\omega_a + 2\omega)^2}$$

$$a_{0,-1} = \left(\frac{\omega_a - \omega}{\omega_a}\right)^2 \qquad a_{32} = \frac{\Lambda_3}{2}$$

$$a_{33} = -1 + \frac{K_x/M}{(\omega_a + 3\omega)^2}$$

The term $-1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{\omega_a^2}$ will be taken as the origin of the determinant.

Define $\Delta_n(\omega_a)$ as the determinant of order $4n-1$ formed by taking a square array from $\Delta(\omega_a)$ centered on the origin. Then

$$\Delta(\omega_a) = \lim_{n\rightarrow\infty} \Delta_n(\omega_a)$$

Define auxiliary determinants from $\Delta_n(\omega_a)$ as follows:

$C_n(\omega_a)$ determinant of order $2n-1$ consisting of the terms to the right of and below the origin term

$D_n(\omega_a)$ determinant of order $2n-2$ obtained from $C_n(\omega_a)$ by omitting its last row and column

$M_n(\omega_a)$ determinant of order $2n-2$ obtained from $C_n(\omega_a)$ by omitting its first row and column

$N_n(\omega_a)$ determinant of order $2n-3$ obtained from $D_n(\omega_a)$ by omitting its first row and column

The determinants $C_n(\omega_a)$ and $D_n(\omega_a)$ satisfy the following recurrence relations:

$$
\left.
\begin{aligned}
C_n(\omega_a) &= \left\{-1 + \frac{K_x/M}{\left[\omega_a + (2n-1)\omega\right]^2}\right\} D_n(\omega_a) \\
&\quad - \frac{\Lambda_3}{2}\left[\frac{\omega_a + (2n-1)\omega}{\omega_a + (2n-2)\omega}\right]^2 C_{n-1}(\omega_a) \\
D_n(\omega_a) &= \left\{-1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{\left[\omega_a + (2n-2)\omega\right]^2}\right\} C_{n-1}(\omega_a) \\
&\quad - \frac{\Lambda_3}{2}\left[\frac{\omega_a + (2n-3)\omega}{\omega_a + (2n-2)\omega}\right]^2 D_{n-1}(\omega_a)
\end{aligned}
\right\}
\qquad (C5)
$$

The recurrence relations (equations (C5)) are also satisfied by $M_n(\omega_a)$ and $N_n(\omega_a)$, with $M$ and $N$ replacing $C$ and $D$, respectively. The initial values are

$$
C_1(\omega_a) = -1 + \frac{K_x/M}{(\omega_a + \omega)^2}
$$

$$
D_2(\omega_a) = \left[1 - \frac{K_x/M}{(\omega_a + \omega)^2}\right]\left[1 - \frac{\omega^2\Lambda_1 + \Lambda_2}{(\omega_a + 2\omega)^2}\right] - \frac{\Lambda_3}{2}\left(\frac{\omega_a + \omega}{\omega_a + 2\omega}\right)^2
$$

$$
N_2(\omega_a) = -1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{(\omega_a + 2\omega)^2}
$$

$$
M_2(\omega_a) = \left[1 - \frac{\omega^2\Lambda_1 + \Lambda_2}{(\omega_a + 2\omega)^2}\right]\left[1 - \frac{K_x/M}{(\omega_a + 3\omega)^2}\right] - \frac{\Lambda_3}{2}\left(\frac{\omega_a + 3\omega}{\omega_a + 2\omega}\right)^2
$$

Expanding $\Delta_n(\omega_a)$ in terms of the elements of the column containing the origin gives

$$\Delta_n(\omega_a) = \left(-1 + \frac{\omega^2\Lambda_1 + \Lambda_2}{\omega_a^{\,2}}\right) C_n(-\omega_a)C_n(\omega_a)$$

$$- \frac{\Lambda_3}{2}\left[\left(\frac{\omega_a - \omega}{\omega_a}\right)^2 C_n(\omega_a)M_n(-\omega_a) - \left(\frac{\omega_a + \omega}{\omega_a}\right)^2 C_n(-\omega_a)M_n(\omega_a)\right]$$

As $n$ becomes infinite, or as $\omega_a$ becomes infinite, the recurrence relations (equations (C5)) approach

$$\left.\begin{aligned} C_n &= -D_n - \frac{\Lambda_3}{2} C_{n-1} \\ D_n &= -C_{n-1} - \frac{\Lambda_3}{2} D_{n-1} \end{aligned}\right\} \tag{C6}$$

Equations (C6) are satisfied by a solution of the form

$$\left.\begin{aligned} C_n &= Ck^n \\ D_n &= -C\left(k + \frac{\Lambda_3}{2}\right) k^{n-1} \end{aligned}\right\} \tag{C7}$$

where $k$ satisfies the equation

$$k^2 - (1 - \Lambda_3)k + \frac{\Lambda_3^{\,2}}{4} = 0 \tag{C8}$$

The larger root of equation (C8) will be denoted by $k$ and the smaller root by $k_1$. Although the complete solution of equation (C6) is of the form

$$C_n = Ck^n + C'k_1^{\,n}$$

for large values of $n$ the term in $k_1$ becomes negligible compared with the term in $k$.

With the same values for $k$, $M_n$ and $N_n$ will have solutions similar to those of equations (C7). Thus as $n$ becomes infinite $\Delta_n(\omega_a)$ will vary as the quantity $k^{2n}$.

Define the function

$$F(\omega_a) = \lim_{n\to\infty} F_n(\omega_a) = \lim_{n\to\infty} \frac{\Delta_n(\omega_a)}{k^{2n}}$$

The function $F(\omega_a)$ is periodic in $\omega_a$ of period $2\omega$, has roots $\pm(\omega_{a_1} \pm 2s\omega)$, $\pm(\omega_{a_2} \pm 2s\omega)$, and second-order poles at $(\omega_a \pm s\omega)$ for all integral values of $s$. Furthermore, $F(\omega_a)$ approaches the limit

$$E = \lim_{\omega_a\to\infty} F(\omega_a) = \lim_{n\to\infty} F_n(\infty)$$

$$= \lim_{n\to\infty} \frac{C_{2n}(\infty)}{k^{2n}} = \frac{k^2}{k_1{}^2 - k^2} ,$$

as $\omega_a$ becomes infinite in a direction other than along the real axis.

Form the function

$$J(\omega_a) = F(\omega_a) \frac{\sin^2\left(\frac{\pi\omega_a}{\omega}\right)}{\left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a_1}}{2\omega}\right)\right]\left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a_2}}{2\omega}\right)\right]} \quad \text{(C9)}$$

The function $J(\omega_a)$ is an analytic function of $\omega_a$ everywhere. Hence, by Liouville's theorem, $J(\omega_a)$ is a constant. By letting $\omega_a \to +\infty$ along the imaginary axis, it is seen that $J(\omega_a) = -4E$.

Substituting $-4E$ for $J(\omega_a)$ into equation (C9) results in

$$\lim_{n\to\infty} \frac{\Delta_n(\omega_a)}{k^{2n}} = F(\omega_a)$$

$$= -4E \frac{\left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)\right]\left[\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) - \sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right)\right]}{\sin^2\left(\frac{\pi\omega_a}{\omega}\right)}$$

Introducing K functions defined similarly to those used for the case $K_y = 0$ gives

$$\left.\begin{aligned} K_1 &= \sin^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)\sin^2\left(\frac{\pi\omega_{a2}}{2\omega}\right) \\ &= \lim_{\omega_a\to 0}\left[\frac{\sin^2\left(\frac{\pi\omega_a}{\omega}\right) F(\omega_a)}{-4E}\right] \\ K_2 &= \cos^2\left(\frac{\pi\omega_{a1}}{2\omega}\right)\cos^2\left(\frac{\pi\omega_{a2}}{2\omega}\right) \\ &= \lim_{\omega_a\to\omega}\left[\frac{\sin^2\left(\frac{\pi\omega_a}{2\omega}\right) F(\omega_a)}{-4E}\right] \end{aligned}\right\} \quad \text{(C10)}$$

Carrying out the limit processes indicated in equations (C10) gives

$$K_1 = \lim_{n\to\infty} K_{1_n}$$

$$= \lim_{n\to\infty}\left[\frac{\pi^2(\omega^2\Lambda_1 + \Lambda_2)C_n^2(0) - \Lambda_3\pi^2\omega^2C_n(0)N_n(0)}{-4\omega^2\ C_{2n}(\infty)}\right]$$

$$K_2 = \lim_{n\to\infty} K_{2_n}$$

$$= \lim_{n\to\infty}\left[\frac{\pi^2\ \frac{K_x}{M}\ kN_n{}^2(-\omega)}{-4\omega^2\ C_{2n}(\infty)}\right]$$

Finally, upon introducing appropriate convergence factors, the quantities needed in equation (36) are

$$K_1 = \lim_{n\to\infty}\left[\frac{K_{1_n}\ \sin^2\left(\frac{\pi R}{2}\right)}{\frac{\pi^2R^2}{4}\prod_{j=1}^{n}\left(1 - \frac{R^2}{4j^2}\right)^2}\right]$$

$$K_2 = \lim_{n\to\infty}\left\{\frac{K_{2_n}\ \cos^2\left(\frac{\pi R}{2}\right)}{\prod_{j=1}^{n}\left[1 - \frac{R^2}{(2j - 1)^2}\right]^2}\right\}$$

where

$$R^2 = \frac{\frac{K_x}{M} + \omega^2\Lambda_1 + \Lambda_2 + \omega^2\left(1 + k_1 - k\right)}{\omega^2\left(k - k_1\right)}$$

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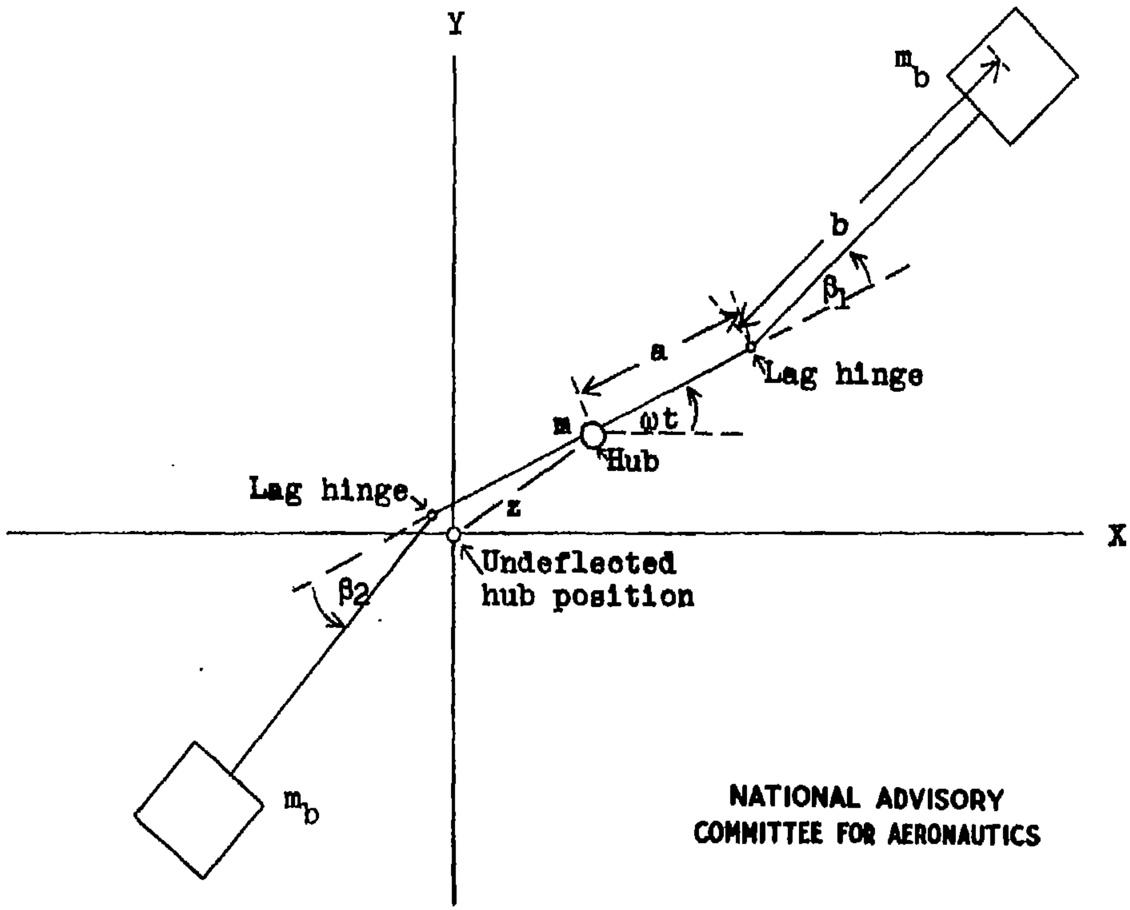


Figure 1.- Simplified mechanical system representing rotor.

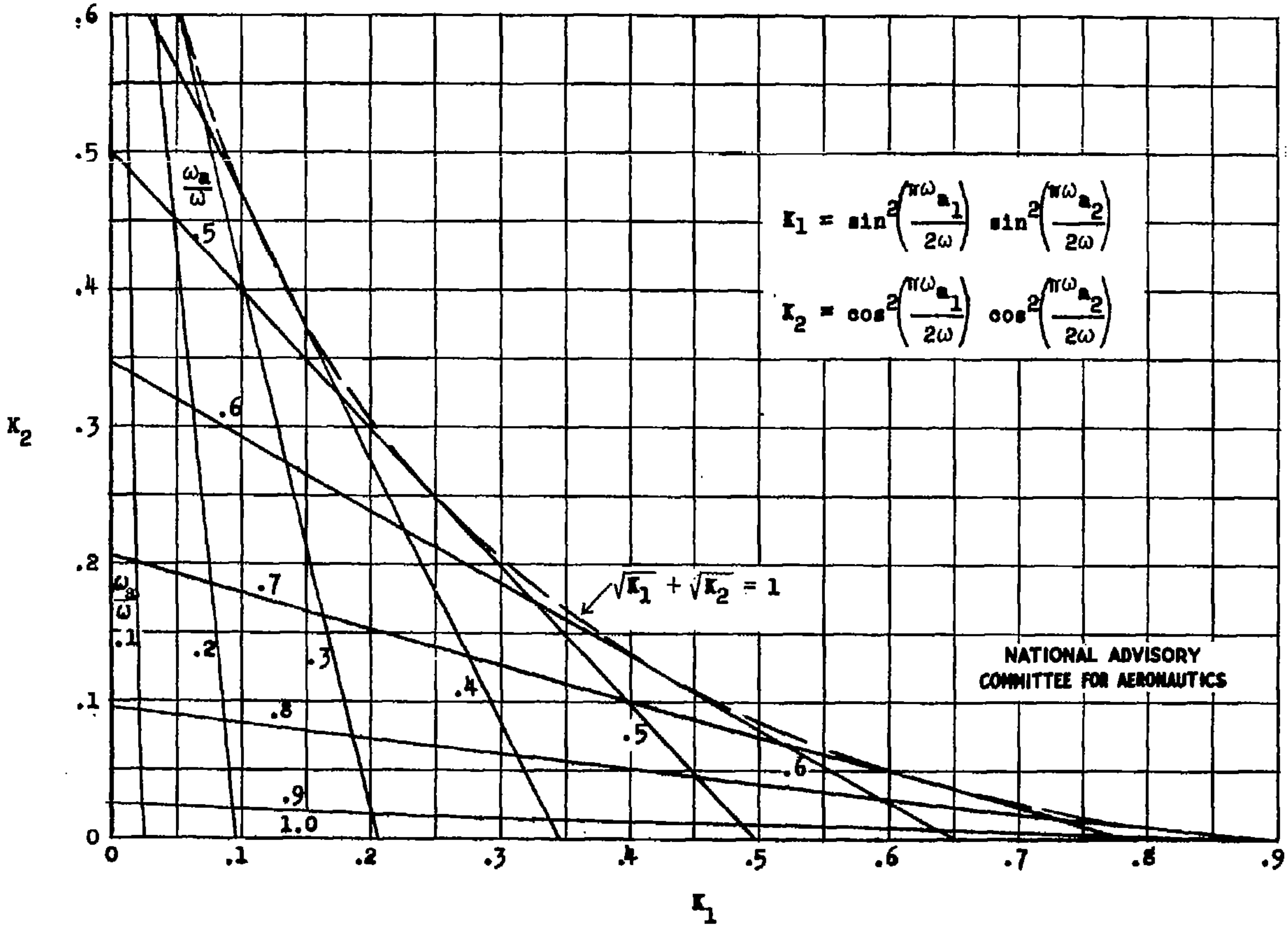


Figure 2.- Chart for obtaining principal value of $\omega_a$ from values of $K_1$ and $K_2$; used in cases of $K_y = 0$ and $K_y = \infty$.

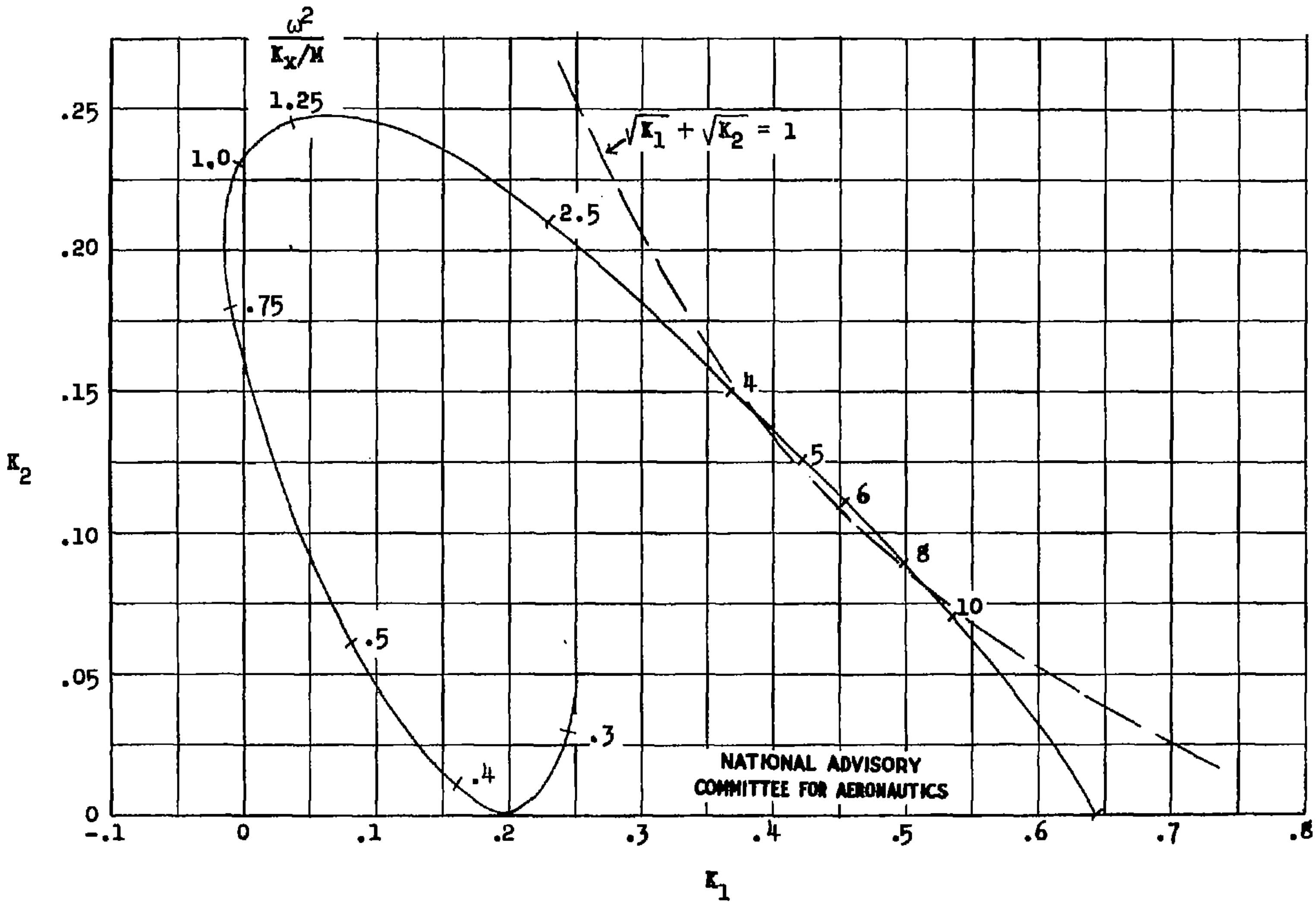


Figure 3.- Graph of $K_1$ and $K_2$ as functions of rotor speed $\omega$ for $\Lambda_1 = 0.1$, $\Lambda_2 = 0$, $\Lambda_3 = 0.1$, and $K_y = 0$.

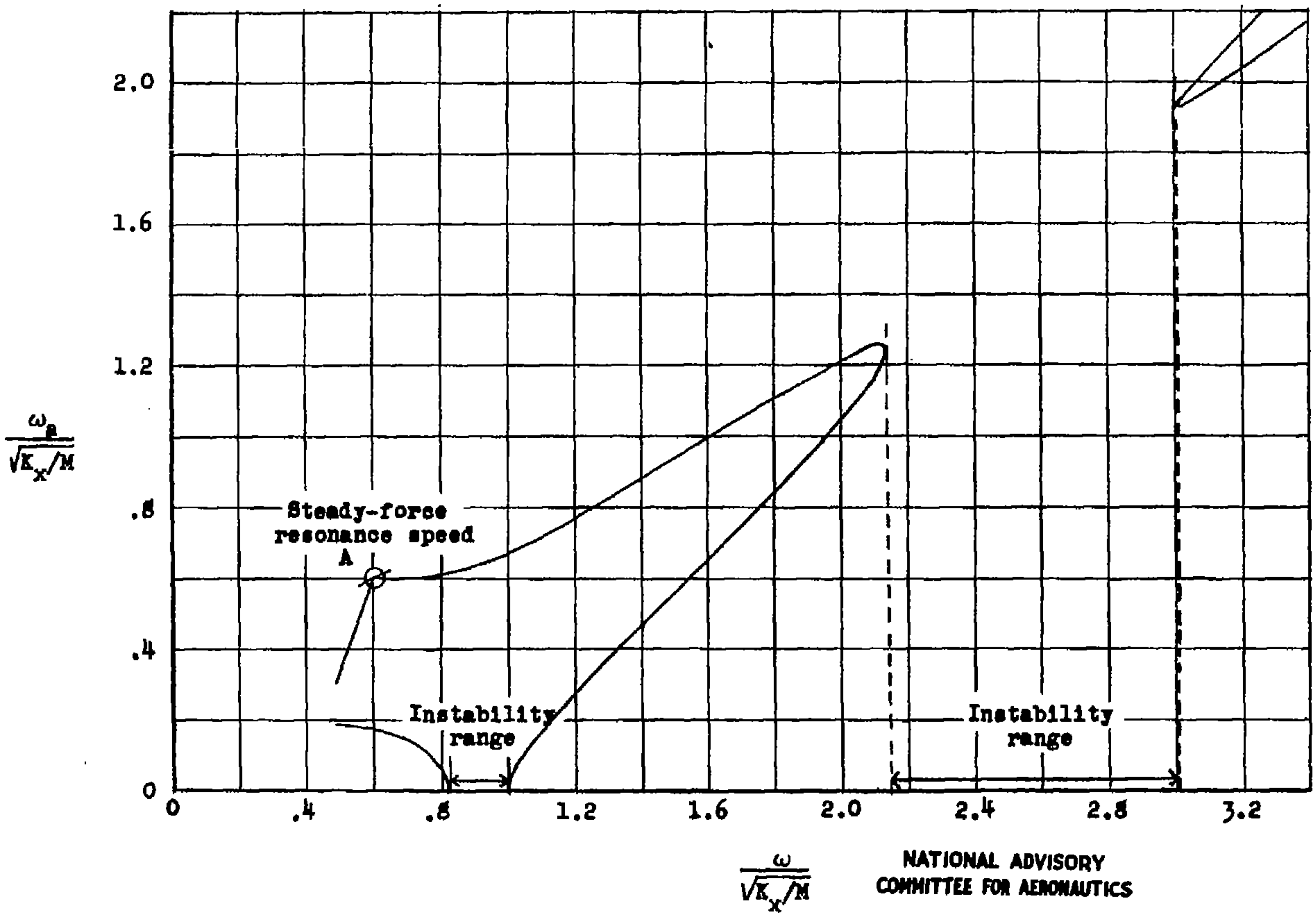


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**Figure 4.- Principal values of $\omega_a$ plotted against rotor speed $\omega$ for case of $A_1 = 0.1$, $A_2 = 0$, $A_3 = 0.1$, and $K_y = 0$.**

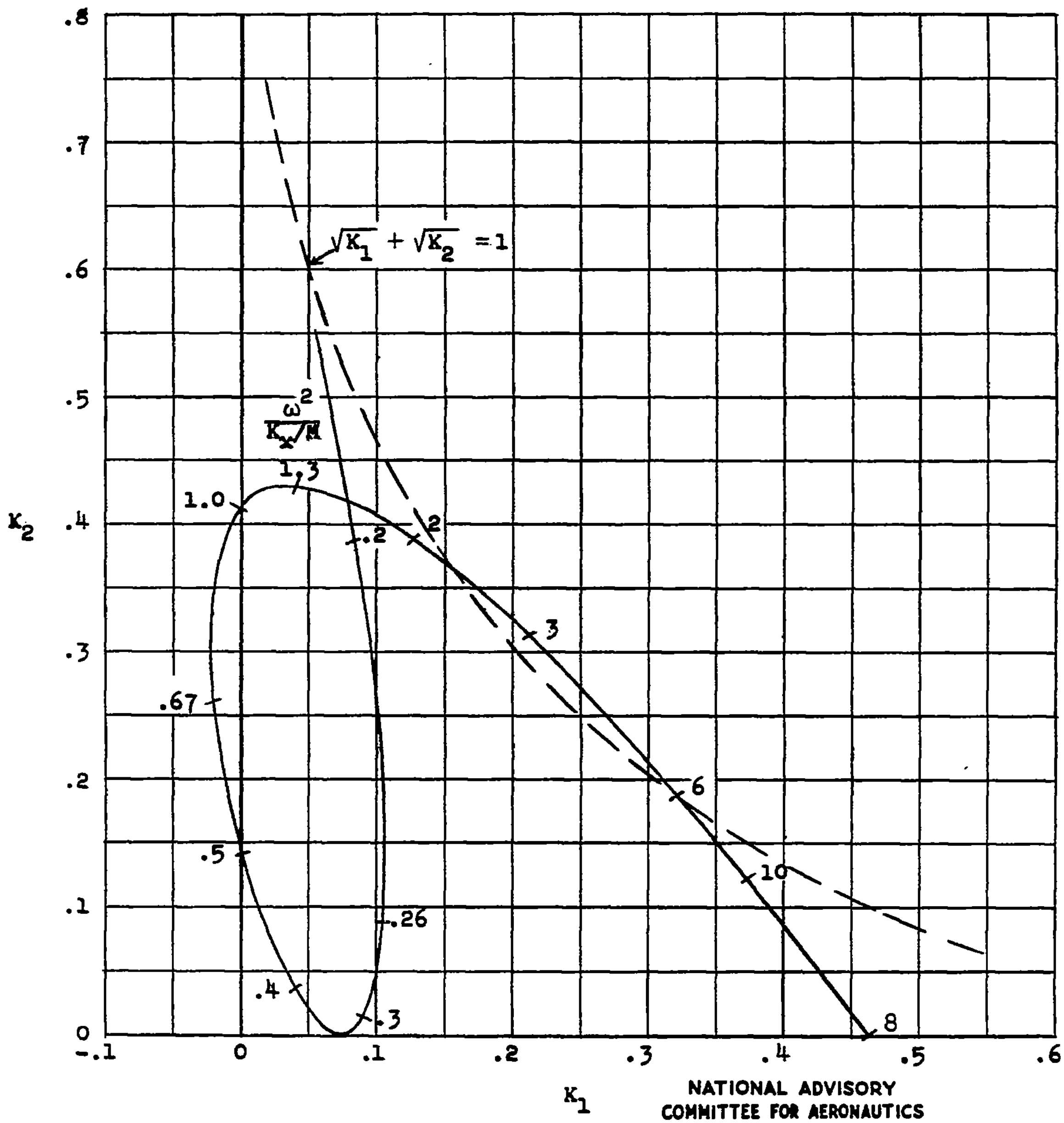


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Figure 5.- Graph of $K_1$ and $K_2$ as functions of the rotor speed $\omega$ for $A_1 = 0.1$, $A_2 = 0$, $A_3 = 0.1$, and $K_y = \infty$.

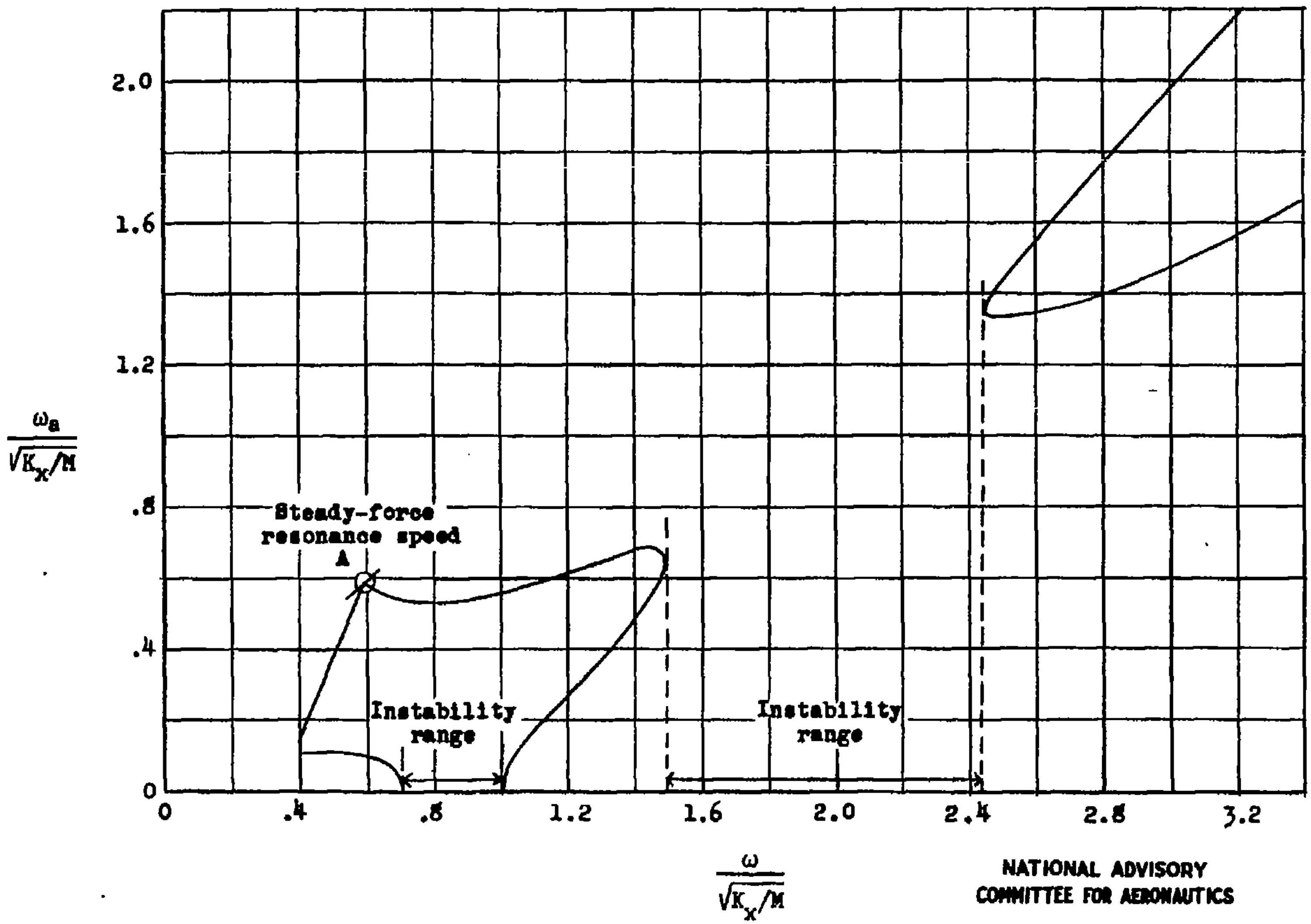




Figure 6.- Principal values of $\omega_a$ plotted against rotor speed $\omega$ for case of $\Lambda_1 = 0.1$, $\Lambda_2 = 0$, $\Lambda_3 = 0.1$, and $K_y = \infty$.

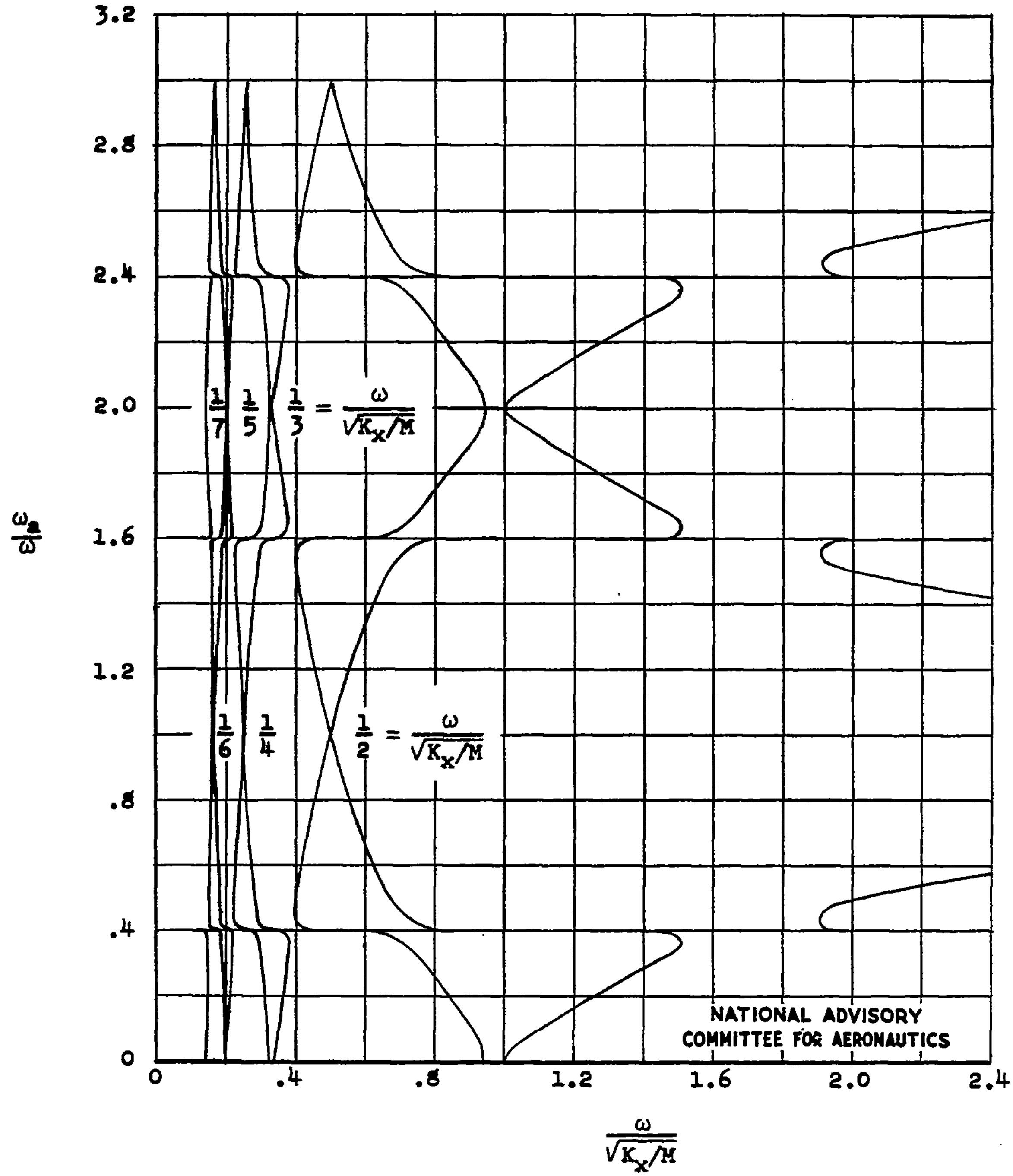


**Figure 7.- Typical pattern of response frequencies against rotor speed $\omega$ for a small value of the mass-ratio parameter $\Lambda_3$.**

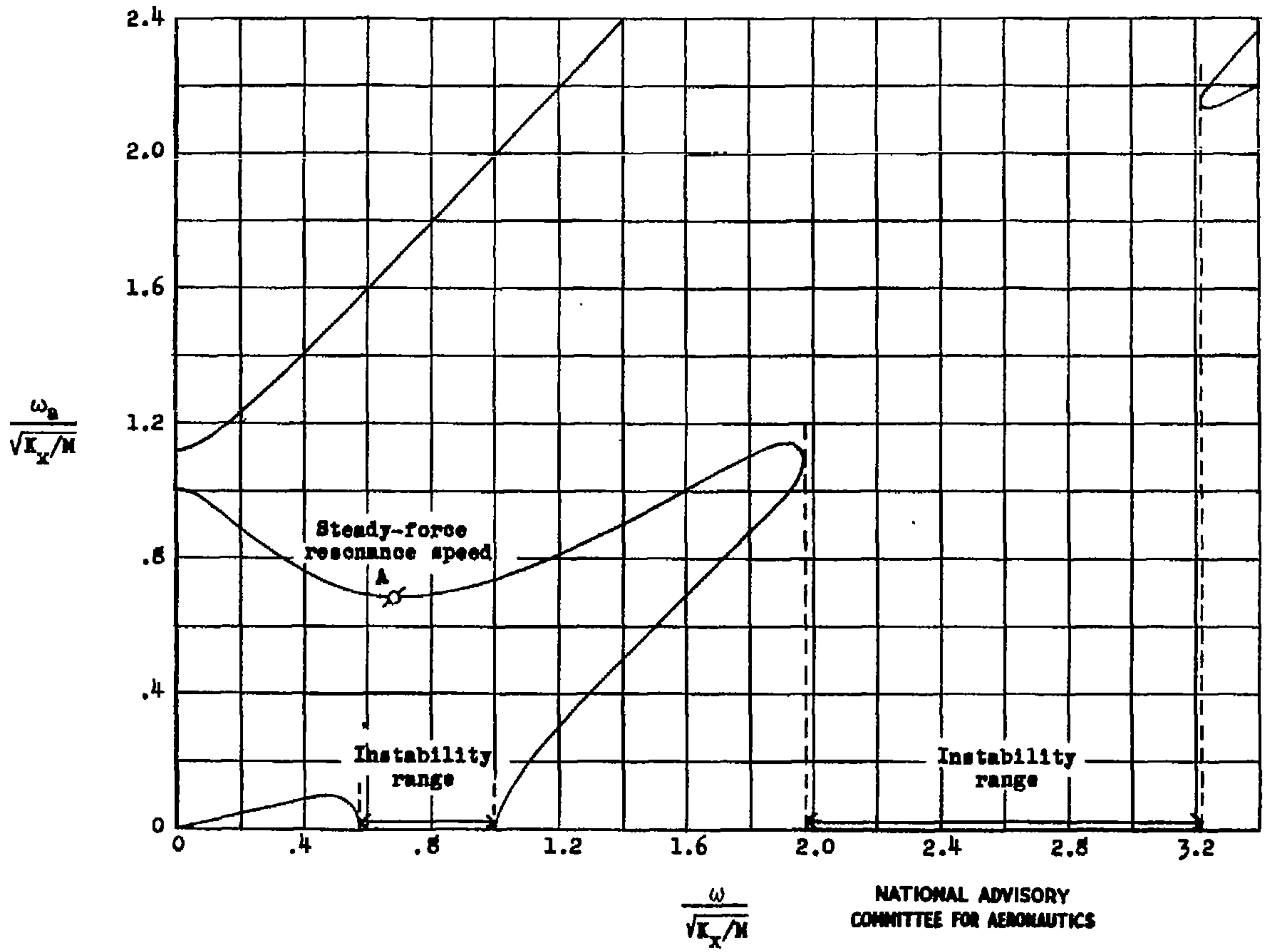


Figure 8.- Response frequencies of a two-blade rotor on symmetric supports for $A_1 = 0.1$, $A_2 = 0$, $A_3 = 0.1$, and $K_y = K_x$.

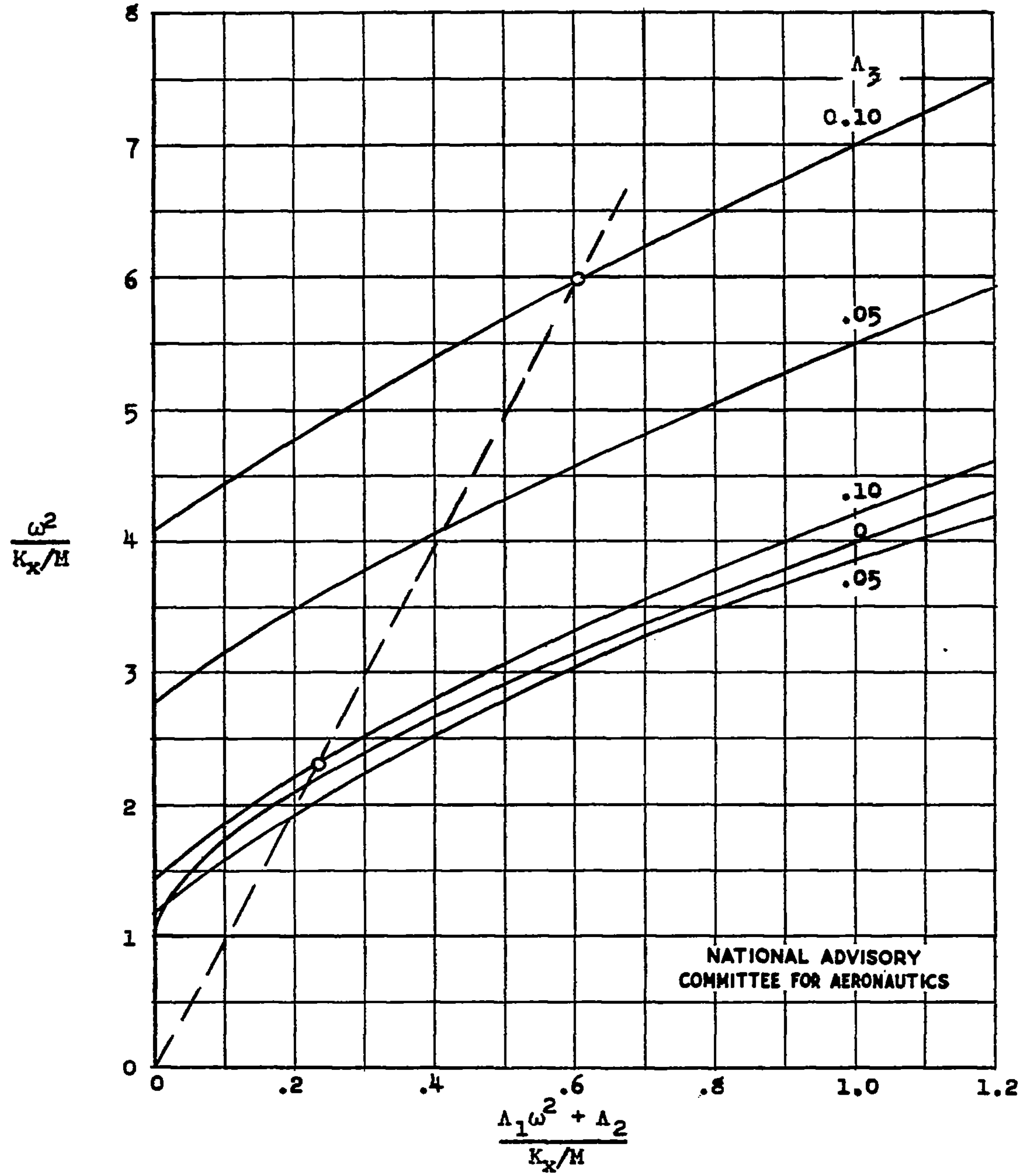


Figure 9.- Chart giving position of the main instability range for $K_y = \infty$.